WAVE DYNAMICS OF STRATIFIED MEDIUMS WITH VARIABLE DEPTH: EXACT SOLUTIONS AND ASYMPTOTIC REPRESENTATIONS

V.V.Bulatov, and Yu.V.Vladimirov,
Institute for Problems in Mechanics RAS
Pr.Vernadskogo 101-1, 119526 Moscow, Russia
bulatov@index-xx.ru

INTRODUCTION.

Wave dynamics of stratified medium (ocean, atmosphere) is highly dependent on bottom topography. The exact analytical solution is obtained only if the water distribution density and bottom shape described by some model functions. When the characteristics of the medium and the boundaries are arbitrary and can be built only numerical solutions of such problems. However, numerical solutions are not qualitatively analyze the characteristics of the wave of the fields. The need for a qualitative analysis of the far field of internal waves arise in the study of internal waves remote methods by means of aerospace-parameter radar. Then the description and analysis of wave dynamics can be made only on the basis of the asymptotic models. In this paper uniform asymptotic forms of the far field of internal gravity waves which propagate in stratified medium with a smoothly varying bottom are constructed

PROBLEM FORMULATION.

In this study we consider a non-viscous incompressible nonhomogeneous medium. If it is unperturbed, we denote its density by $\rho(z)$ (the stratification is supposed to be stable, i.e. $\partial\rho/\partial z<0$, the axis z is directed downward from the medium surface). The system of the hydrodynamic equations takes the following form [1,2]

$$\frac{\partial \mathrm{v}}{\partial t}+(\mathrm{v}\cdot\nabla)\,\mathrm{v}=-\frac{\nabla p}{\rho}+\mathrm{g}\,,\ div\,\mathrm{v}=0\,,\ \frac{\partial\rho}{\partial t}+\nabla\rho\cdot\mathrm{v}=0 \qquad (1)$$

where $\mathrm{v}=(U_1,U_2,W)$ is the velocity vector, $\mathrm{g}=(0,0,g)$ is the gravitation acceleration vector, p and ρ are the deviations of the pressure and the density from their equilibrium values. We consider a stratified medium with density $\rho_0(z)$, bounded in unperturbed state by the surface $z=0$ and bottom $z=-H(y)$, with its depth depending solely on one horizontal variable y.

EXACT AND ASYMPTOTIC SOLUTIONS.

Then, restricting ourselves to the case of constant Vaisala-Brunt frequency $N^2(z)=-\dfrac{g}{\rho_0(z)}\dfrac{d\rho_0(z)}{dz}=const$, time law $\exp(-i\omega t)$ and the dependence of x in the form of $\exp(ilx)$ in the Boussinesq approximation we shall obtain the following linearized about the quiescent state equation for the vertical components of velocity $W(z,y)$ (omitting the multiplier $\exp(ilx)$)

$$\frac{\partial^2 W}{\partial^2 z^2}-\frac{N^2-\omega^2}{\omega^2}\frac{\partial^2 W}{\partial^2 y^2}+l^2\frac{N^2-\omega^2}{\omega^2}W=0 \qquad (2)$$

As the boundary conditions we take the “rigid lid” condition at $z=0$ and no-fluid-loss condition at the bottom: $W=0$ at $z=0$, $W+\dfrac{dH(y)}{dy}U_2=0$ at $z=-H(y)$ where (U_2,W) - velocity components. With relation to the function $H(y)$ we assume as follows: $H(y)$ is the continuously differentiable function with no more than a single minimum. The continuity variation $H(y)$ means that the relationship between the horizontal scale L of variation $H(y)$

and the vertical scale M is defined by the value $\lambda = L/M >> 1$. In non-dimensional variables $x^* = x/L$, $y^* = y/L$; $z^* = z/M$, $h^*(y^*) = H(y)/M$; $l^* = lM$, $\omega^* = \omega/N$ the equation (2) and the boundary conditions are re-written as follows (the sign * is further-on omitted)

$$\frac{\partial^2 W}{\partial^2 z^2} - \frac{1}{\lambda^2 c^2}\frac{\partial^2 W}{\partial^2 y^2} + \frac{l^2}{c^2} W = 0$$

$$W = 0 \text{ , at } z = 0 \qquad W + \frac{dh(y)}{\lambda dy} U_2 = 0 \text{ , at } z = -h(y),\ c^2 = \frac{\omega^2}{1-\omega^2} \tag{3}$$

Below we substitute the boundary condition at the bottom with large λ by $W = 0$ at $z = -h(y)$. We shall seek the asymptotic solution of the problem (3.6.3) in the form typical for the geometric optics (WKBJ) method [3,4]

$$W = \left(F_0(z,y,\omega) + \frac{i}{\lambda} F_1(z,y,\omega) + \left(\frac{i}{\lambda}\right)^2 F_2(z,y,\omega) + ... \right) \exp(i\lambda S(y,\omega)) \tag{4}$$

where $F_m(z,y,\omega) = 0$ at $z = 0$ and at $z = -h(y)$, $m = 0,1,2,...$ Substituting the solution for W in the form (4) into the equation (3) and making equal the members at λ^0 and λ^1, we obtain

$$\frac{\partial^2 F_0}{\partial^2 z^2} + \frac{S'^2 + l^2}{c^2} F_0 = 0,\ F_0 = 0 \text{ , at } z = 0 \text{ and at } z = -h(y)$$

$$\frac{\partial^2 F_1}{\partial^2 z^2} + \frac{S'^2 + l^2}{c^2} F_1 = \frac{1}{c^2}\left(2F_0' S' + F_0 S''\right),\ F_1 = 0 \text{ , at } z = 0 \text{ and at } z = -h(y) \tag{5}$$

where prime marks without indexes denote the derivatives over y. The solution of the first equation from (5) (the Sturm-Liouville problem) provides for the mode structure solution: the dispersion relationships $\kappa_n^2(y,\omega) = \dfrac{c^2 n^2 \pi^2}{h^2(y)}$, $n = 1,2,...$ and the eigenfunctions in a zero-order approximation (vertical modes) $F_{0n}(z,y,\omega) = A_{0n}(y,\omega)\sin(n\pi z/h(y))$, $n = 1,2,...$. The eikonal $S_n(y,\omega)$ is defined from the relationship $\kappa_n^2(y,\omega) = S_n'^2(y,\omega) + l^2$. To find the amplitude $A_{0n}(y,\omega)$ we use the resolvability condition for the second equation from (5), which requires the orthogonality of the equation's right member and of the function F_{0n}. By multiplying at fixed n this equation by F_{0n} and integrating over z from 0 to $h(y)$, we obtain the following «conservation law» $\dfrac{\partial}{\partial y}\left(\int_0^{h(y)} F_{0n}^2(z,y,\omega)dz \cdot S'(y,\omega)\right) = 0$. By integrating the eigenfunction and considering the equation for the eikonal we finally get: $A_{0n} = \dfrac{B_{0n}(y_0,\omega)}{\sqrt[4]{c^2 n^2 \pi^2 - h^2(y) l^2}}$, where the variable B_{0n} depends on ω and the initial eikonal value at any point y_0, $S_n(y_0,\omega)$. The eikonal $S_n(y,\omega)$ shall be defined as $S_n(y,\omega) = \int_y^{y^*} \sqrt{\kappa_n^2(y,\omega) - l^2 dy}$, where y^* is the «turning point», i.e. the root of equation $\kappa_n^2(y,\omega) = l^2$. Then WKBJ solution for a separate wave mode is given as follows

$$W_n^{\pm}(z,y)=\frac{D_{\pm}}{\sqrt[4]{c^2n^2\pi^2-h^2(y)l^2}}\exp(\pm i\lambda(S_n(y,\omega)))\sin\frac{n\pi z}{h(y)}$$

where plus-sign corresponds to an «incident» wave, and the minus-sign corresponds to a reflected wave. A geometric solution is not working near the turning point (the amplitude A_{0n} tends to infinity).The solution's uniform asymptotics for a individual wave mode applicable at the turning point is expressed by the Airy function and is given in the form

$$W_n=\frac{\sqrt{2}\pi\left(\frac{3}{2}\lambda S_n(y,\omega)\right)^{1/6}}{\lambda^{1/2}\sqrt[4]{c^2n^2\pi^2-h^2(y)l^2}}Ai\left(\left(\frac{3}{2}\lambda S_n(y,\omega)\right)^{2/3}\right)\sin\frac{n\pi z}{h(y)}$$

where $Ai(x)$ is the Airy function. The complete solution in the WKBJ approximation for a single mode before the turning point (i.e. within the «wave region») appears to be

$$W_n=\frac{\sqrt{2\pi}}{\lambda^{1/2}\sqrt[4]{c^2n^2\pi^2-h^2(y)l^2}}\cos\left(\lambda S_n(y,\omega)-\frac{\pi}{4}\right)\sin\frac{n\pi z}{h(y)}$$

and beyond the turning point (in the region of exponential fading)

$$W_n=\frac{\sqrt{\pi}}{\lambda^{1/2}\sqrt[4]{c^2n^2\pi^2-h^2(y)l^2}}\exp(-\lambda|S_n(y,\omega)|)\sin\frac{n\pi z}{h(y)}$$

For a linear bottom profile this problem we solved analytically. The solution for a individual wave mode is given by the Macdonald function K_ν of imaginary index as follows

$$W_n=e^{-i\frac{\pi\nu}{2}}K_\nu\left(l\sqrt{\lambda^2y^2-\frac{z^2}{c^2}}\right)\sin\left(\frac{n\pi}{\ln\Delta}\ln\frac{\lambda cy-z}{\lambda cy+z}\right)$$

where $\Delta=\frac{\lambda c+1}{\lambda c-1}$, $\nu=\frac{2\pi ni}{\ln\Delta}$, $c^2=\frac{\omega}{1-\omega^2}$, $\lambda=\frac{1}{\gamma}$ γ is the bottom's inclination, $h(y)=-y$.

NUMERICAL RESULTS AND DISCUSSIONS.

The figures show the results of internal gravity wave vertical velocity calculations for two non-linear stratified medium bottom shape. Numerical calculations show that the two types of existent waves: captured waves and progressive waves. In fig.1 results of a function $W_1(y,z)$ calculations are presented. In fig.2 results of a function $W_2(y,z)$ calculations are presented. The stratified medium bottom shapes are shown in figures. The effect of spatial-frequency "blocking" of the wave field was revealed. Depending on the wave and bottom shape the far internal waves field can be localized in a bounded region of space (captured waves), or spread, in the absence of turning points, over a large distance compared to the medium depth (progressive waves). The region, which progressive wave can penetrate, is completely determined by the presence of turning points, whose locations depend on the medium stratification and bottom topography. In this paper asymptotic and exact representations of solutions was obtained that describe the far field of internal gravity waves in a stratified mediums of variable depth. Using developed asymptotic methods, one can consider a wide class of interesting physical problems, including problems concerning the propagation of internal gravity wave packets in non-homogeneous stratified media under the assumption that the modification of the parameters of a vertically stratified medium are slow in the horizontal direction. Numerical analyses that are performed using typical ocean parameters reveal that actual dynamics of the internal gravity waves are strongly influenced by horizontal non-homogeneity of the ocean bottom. In this paper we use an analytical approach, which avoids the numerical calculation widely used in analysis of internal gravity wave dynamics in stratified mediums. Asymptotic solutions, which are obtained in this paper, can be of significant importance for engineering applications, since the method of geometrical optics, which we modified in order to calculate the wave field near caustic, makes it possible to describe different wave fields in a rather wide class of other problems.

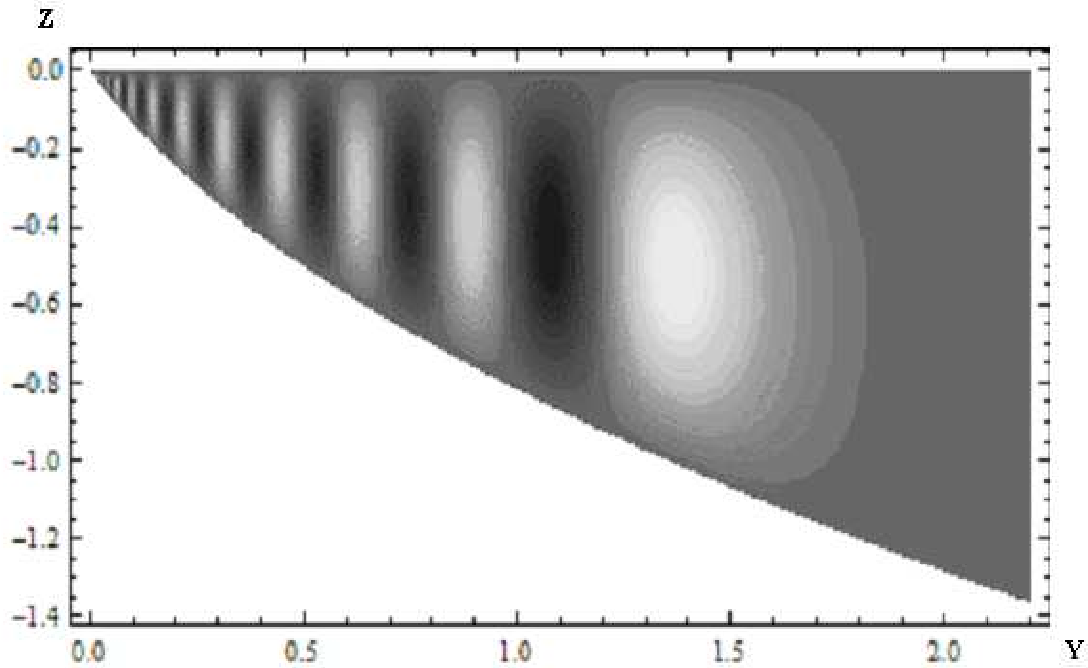

Fig.1 Internal gravity waves first mode for infinitely descending bottom profile (captured waves)

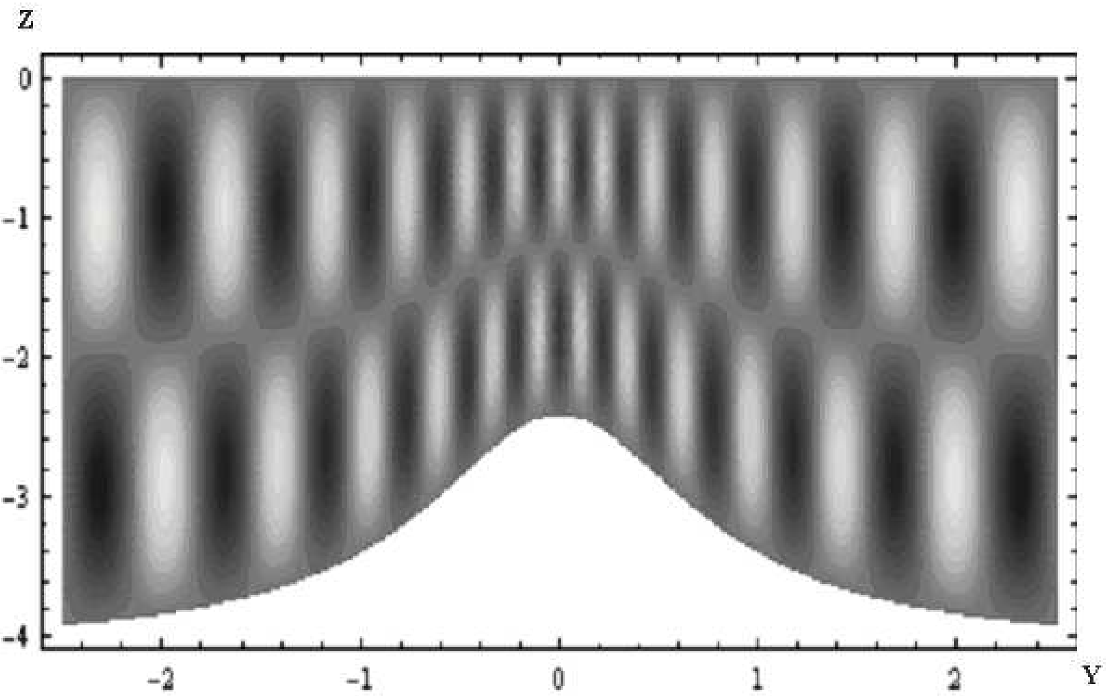

Fig.2 Internal gravity waves second mode for hill bottom profile (progressive waves).

ACKNOWLEDGMENTS

The results presented in the paper have been obtained by research performed under projects supported by the Russian Foundation for Basic Research (No.11-01-00335a "Stability and Instability of Flows with Interfaces", No. 12-05-00227a “Dynamics of Wave Processes in Conditions of the Arctic Basin”), Program of the Russian Academy of Sciences “Fundamental Problems of Oceanology: Physics, Geology, Biology, Ecology” .

REFERENCES

1. *V.V.Bulatov, Yu.V.Vladimirov* Internal gravity waves: theory and applications. - Moscow: Nauka Publishers, 2007.
2. *V.V.Bulatov, Yu.V.Vladimirov* Wave dynamics of stratified mediums. - Moscow: Nauka Publishers, 2012.
3. *V.V.Bulatov, Yu.V.Vladimirov* Far fields of internal gravity waves in stratified media of variable depth // Rus. J. Math. Phys., 2010. V. 17, P. 400-412.
4. *V.V.Bulatov, Yu.V.Vladimirov* The uniform asymptotic form of the internal gravity wave field generated by a source moving above a smoothly varying bottom // J. Eng. Math., 2011, V.69. P. 243-260.