Fundamental Problems of Internal Gravity Waves Dynamics in Ocean

Vitaly V. Bulatov* and Yury V. Vladimirov

Institute for Problems in Mechanics RAS, Pr. Vernadskogo 101 - 1, Moscow 119526, Russia

Abstract: In paper fundamental problems of internal gravity waves dynamics are considered. The solution of this problem is expressed in terms of the Green's function and the asymptotic representations of the solutions are considered. The uniform asymptotic forms of the internal gravity waves in horizontally inhomogeneous and non-stationary stratified ocean are obtained. A modified spatio-temporal ray method is proposed, which belongs to the class of geometrical optics methods (WKBJ method). Analytical and numerical algorithms of internal gravity wave calculations for the real ocean parameters are presented.

Keywords: Stratified ocean, internal gravity waves, asymptotic methods.

INTRODUCTION

Internal gravity waves are an important feature of ocean dynamics and the may be found almost everywhere in the ocean. Internal gravity waves are supported in a stably stratified medium. Vertical motion is then suppressed by gravitational forces. A vertical displacement of a fluid parcel will result in a buoyancy force opposite in direction to the displacement and as a result the fluid parcel will undergo vertical oscillations with a frequency which is determined by the density stratification of the medium. The essential characteristic of internal gravity waves is that they can propagate over large distance practically without dissipation losses and with almost unchanged structure. There are many works devoted to theoretical, numerical and experimental investigations of the internal gravity waves fields, but our aim is not to discuss this results. The reader is referred to, among others, the papers [1-19].

There are several types of energy source for internal gravity waves. One possibility is generation through atmospheric winds. Another is tidal flow over bottom topography. To generate an internal wave field with an organized structure it is necessary to have a coherent localized source. One example of this is a moving vessel. Both surface and underwater vessels generated internal gravity waves which have a well organizes structure. Just as for surface ship waves, the internal wave field has a V-shaped structure, but in contradistinction to ship waves here the opening angle oh the V depends on the stratification and the speed of the vessel. A self-propelled moving underwater object will create a turbulent wake. The wake will form behind the body and it will expand in time, both horizontally and vertically. Further this wake collapses at a certain distance downstream of the moving object and produces a significant internal wave field. The numerical modeling of such wake collapse is difficult, and requires a large amount of computer capacity. Such a model is outside the scope of our investigation [14, 15].

For an arbitrary ocean density distribution it is possible to solve the internal wave generation and propagation problems numerically. The shortcoming of this approach is the boundness of the space region in which the problem can be solved. For example, numerical Fourier method is used usually. In accordance with this method it is necessary to sum over about 10^6 Fourier components. Moreover, the numerical method do not readily lead to a qualitative description of internal waves in real ocean. In this paper we will examine the linear internal gravity waves generated by u source in stratified medium. This problem is solved by using Green's function method. The solution thus obtained is a sum of triple quadratures which expressed in terms of eigenfunctions and eigenvalues of corresponding vertical spectral problems. This method makes it possible to obtain the simple asymptotic forms of solution and easy-to-interpret qualitative description of field structure [6, 9, 11, 12].

The internal gravity waves are the oscillations of a stratified medium in the gravity force field. The stratified medium is such a medium where the density increases with the depth. Suppose that a volume element of the medium is not at the equilibrium, for example it could be displaced upward, then it will be heavier than the surrounding medium and therefore Archimedean forces will make it move back to the equilibrium. The essential parameter of any oscillating system is the frequency. It is determined by the correlation of two factors:

*Address corresponding to this author at the Institute for Problems in Mechanics RAS, Pr. Vernadskogo 101 - 1, Moscow 119526, Russia; Tel: +7-903-7220761; Fax: +7-499-739-9531; E-mail: bulatov@index-xx.ru

ISSN: 1814-8085 / E-ISSN: 1927-5129/13

returning forces which return the perturbed system towards its equilibrium and the inertial forces. For the internal gravity waves the returning forces are proportional to the vertical gradient of the fluid's density and the inertial ones are proportional to the density itself. For the characteristic frequency of the gravity waves oscillations we have the following expression: $N^2(z) = -gd \ln \rho(z)/dz$. This frequency is usually called by the Brunt-Vaisala frequency or the buoyancy frequency. Here $\rho(z)$ is the density considered as a function of the depth z, g is the acceleration in the gravity force field, the sign "-" originates from the increase of the density with the depth and therefore $d\rho(z)/dz < 0$. The exact solutions of the essential equations describing the internal gravity waves are only obtained for special cases. That is the reason why the approximate asymptotical methods are systematically used for the investigation of the internal gravity wave fields in stratified ocean. The internal gravity waves are usually represented in the following integral form: $J = \int\limits_{\gamma} \exp[\lambda f(\tau)] F(\tau) d\tau, \quad \lambda >> 1$, where $f(\tau)$ and $F(\tau)$ are analytic functions of the complex variable τ, γ is a contour of integration on the complex plane τ. The universal way to construct the asymptotic forms of such integrals is the method of etalon integrals [20-23].

This paper is devoted to the description of a generalization of the geometrical optics method (WKBJ method), i.e. we discuss the spatio-temporal ray method of etalon functions. This method allows one to solve the problem of asymptotic modeling of the inharmonic wave packet's dynamics for the internal gravity waves in stratified media with slowly varying parameters. The main reasons to use the ray methods are the following: the ray representations are well correlated with the intuition and with the empirical material for the propagation of the internal gravity waves in natural stratified media (ocean, atmosphere). These methods are universal and very often one can use only them for the approximate computations of the wave fields in slowly changing non-homogeneous stratified media [14, 15, 17].

The horizontal non-homogeneity and non-stationarity are crucial for the propagation of the internal gravity waves in natural stratified media (such as the ocean and the atmosphere). To the most typical horizontal inhomogeneities of the real ocean we relate the change in the ocean bottom shape, the inhomogeneity of the density field and the variance of the mean currents. The exact solution of the problem, for example by means of separating of variables, can only be obtained when the density distribution and the ocean bottom shape are described by the simple model functions. For the arbitrary stratification and the arbitrary ocean bottom topography it is only possible to construct the asymptotic representations of the solutions [16, 17, 19].

However, if the depth or the ocean and its density vary slowly in comparison with the characteristic length (period) of the internal gravity waves, which takes place in the real ocean, then one can use the spatio-temporal ray method (the geometrical optics method, WKBJ method) and its generalizations to investigate the mathematically modeled dynamics of the internal gravity waves. This is the method of etalon functions. Its distinguishing feature is that in order to investigate the evolution of non-harmonic wave packets in stratified non-stationary horizontally non-homogeneous media one seeks for the solution in form of rational powers series with respect to the small parameter. The powers depend on the form of representation for the wave packet. The form of representation is determined by the asymptotic behavior of the solution in the stationary horizontally homogeneous case. The phase of the wave packet can be obtained from the corresponding eikonal equation, which can be solved numerically on the characteristics (rays). The amplitude of the wave packet can be found from a conservation law along the characteristics (rays) [14, 15, 20, 21].

The slowness condition of the change in parameters of the medium in time and along the horizontal is crucial for applying the geometrical optics methods. The slowness is considered in comparison with the characteristic lengths and periods of internal gravity waves. However, these conditions are not sufficient for the geometrical optics methods to be valid. It is clear that for the estimates of the accuracy of the geometrical optics method one has to use the results obtained by a more precise approach than that of the spatio-temporal ray method. However because of the serious mathematical difficulties it is not yet possible. For the investigation of the dynamics of inharmonic internal gravity wave packets in stratified non-homogeneous and non-stationary media we have at hand the analytic methods which are limited and do not allow one to estimate the accuracy of the geometrical optics method for the real media. In the general case there are no exact solutions, and the known rigorous solutions just indicate a possible value of inaccuracy for typical cases. The same results for the value of inaccuracy of the spatio-temporal ray method can be obtained comparing the asymptotic results with the approximate, but more general than that of the ray method, solutions of the basic wave problems. Therefore the validity of the spatio-temporal method and of its results follows from the comparison of the results with the data of natural experiments [5, 12, 13,16].

1. PROBLEM FORMULATION

1.1. Wave Dynamics in Vertically Stratified Mediums

Generally the system of the linear equations describing the small movements of the originally quiescent incompressible non-viscous stratified medium in the system of the Cartesian coordinates *(x,y,z)* with the axis *z* directed vertically upwards, looks like [14-18]

$$divU = Q(x,t)$$

$$\rho_0 \frac{\partial U}{\partial t} + gradp + F = S(x,t) \tag{1}$$

$$\frac{\partial \rho}{\partial t} + \frac{d\rho_0}{dz} W = K(x,t)$$

where $U = (U_1, U_2, W)$, p, ρ - perturbation of the velocity vector, pressure and density; ρ_0 (z) – stratified medium density in the quiescent state; $F=(0,0,g\rho)$, *g*- acceleration of the gravity. Functions *Q,S,K* represent intensities of distributions of the sources of weight, pulses and density accordingly. Boundary conditions on the free surface $z = 0$ and on the flat bottom $z = -H$ look like

$$W = \partial\eta / \partial t \quad \text{p-g}\rho_0\, \eta = \text{P(x,y,t)} \quad z = 0 \tag{2}$$

$$W = Z(x,y,t) \quad z = -H$$

Here function η (x, y, t) describes the vertical displacement of the free surface; P - external pressure, acting on the free surface; and Z – the vertical speed of the bottom. The initial conditions at t=0 are as follows:

$$U = U^*(x),\ \rho=\rho^*(x),\ \eta=\eta^*(x, y) \tag{3}$$

where functions $U^*(x)$, ρ*, η* - initial values of generations of the vector of speed, density and elevation of the free surface. To ensure the correct performance of the condition it is required to meet the following condition: div $U^*(x)$ = Q ($t = 0$) (t=0).

By virtue of the linearity of the problem the forced waves are represent by the superposition of the free harmonious waves described by the homogeneous system (1) and the homogeneous boundary and initial conditions of (2, 3). The system (1) can be reduced to one equation for any of required functions, usually it is done for the vertical velocity component. At that the homogeneous system (1) and the homogeneous boundary conditions (2) may be presented in the form

$$\frac{\partial^2}{\partial t^2}\left((\frac{\partial^2}{\partial z^2} + \Delta)W - \frac{N^2(z)}{g}\frac{\partial W}{\partial z} \right) + N^2(z)\Delta W = 0$$

$$\frac{\partial^3 W}{\partial z\, \partial t^2} - g\Delta W = 0, z = 0 \tag{4}$$

$$W = 0, -H$$

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},\ N^2(z) = -g\frac{d\rho_0}{\rho_0 dz}$$

The first equation from (4) is to some extend simplified after introduction of Boissinesq approximation. At usage of this approximation in the equations of the pulse preservation (1) the difference of the density from some constant value ρ_s, is considered only in the member describing floatability, in the inertial members the real density is replaced with the value ρ_s, and the equation (4) is reduced to the kind

$$\frac{\partial^2}{\partial t^2}(\frac{\partial^2}{\partial z^2} + \Delta)W + N^2(z)\Delta W = 0 \tag{5}$$

The function $N(z)$ is one of the basic characteristics of the stratified medium, and has the fundamental value in the theory of the internal gravity waves and is called the buoyancy frequency or Vaisala-Brunt frequency. The value T=2π/N defines Vaisala-Brunt period. For the real ocean and the atmosphere the value T varies from minutes up to several hours, and for the stratified liquid produced in the laboratory, it can make some seconds (Figure **1**).

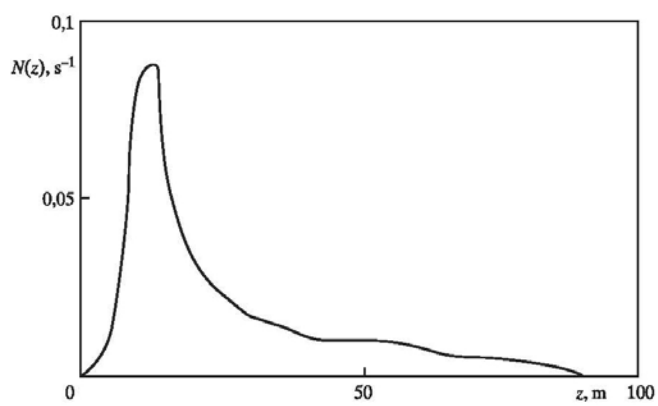

Figure 1: Buoyancy frequency (Vaisala-Brunt frequency) *N(z)* distribution in real ocean.

Homogeneity of the equations (4, 5) and their boundary conditions at the variables x, y, t allow to look for the elementary wave solutions in the field of the plane waves: $W(x,t) = \phi(z)\exp(ikr - i\omega t)$, where *k* is the wave vector in the plane *x, y*; ω - oscillations frequency; *r = (x, y)*.

For function $\phi(z)$ from (4) the boundary problem results in the following Sturm-Liouville equation

$$\frac{\partial^2\phi}{\partial z^2}-\frac{N^2(z)}{g}\frac{\partial\phi}{\partial z}+\left(\frac{N^2(z)}{\omega^2}-1\right)k^2\phi=0 \quad (6)$$

$$\frac{\partial\phi(0)}{\partial z}=g\kappa^2\phi(0)/\omega^2,\ \phi(-H)=0$$

in Boissinesq approximation

$$\frac{\partial^2\phi}{\partial z^2}+\left(\frac{N^2(z)}{\omega^2}-1\right)\kappa^2\phi=0 \quad (7)$$

where k = | k |. Problems (6, 7) are the problems of the own values, after solution of which, there may be defined the system of the own values ω (dispersive dependences) and own functions $\phi(z)$ for each fixed value of the wave number k. The spectrum of such problems is always discrete, that is the system possesses the countable number of the modes $\phi_n(z)$ (n = 1,2,3...), to each of which there corresponds the law of the dispersion $\omega_n=\omega_n(\kappa)$. In the case when the depth of the liquid is endless and the difference of the function N (z) from zero takes place also within the unlimited interval, then alongside with the discrete spectrum there is also a continuous spectrum.

The knowledge of the dispersive dependences and their properties has the paramount value at research of the linear gravity waves. The basic properties of the own values and the own functions of the problems (6, 7) are well studied. The own functions of the considered problems may be divided into to two classes. The first class is presented by one own function $\phi_0(z)$, which is monotonically and quickly enough decreasing with the increasing depth. This own function poorly depends on the conditions of stratification and describes the superficial wave. All other own functions correspond to the normal modes of the internal waves. For the internal waves own function $\phi_n(z)$ (n=1,2,3...) has n-1 zero inside of the interval [-H, 0]. For the continuously stratified liquid of the final depth both for its superficial wave and for its internal waves is typical the monotonous increase in frequency ω of a single mode at the growth of the wave number k, the monotonous reduction of the phase speeds $c_f=\omega/\kappa$ with the growth of k and at the increase of the mode number, and also excess of the phase speeds over the group speeds $c_g=d\omega/d\kappa$. The maximal values of the phase and the group speeds coincide and take place at k = 0. The significant difference of the superficial wave from the internal waves consists that in the short-wave region (κ→oo) the frequency of the superficial wave is unrestrictedly increasing ($\sim\kappa^{1/2}$), whereas the internal waves frequency tends to the value $\max_z N(z)$.

Rather small change of the medium density at changing the depth in comparison with the drop of the density on the water – air border allows to research the internal waves in the approximation of the "solid cover" ($\phi(0)=0$), which filters the superficial waves out without essential distortion of the internal waves. Approximation of "the solid cover" allows to neglect the first sum component in the dynamic condition of (2).

The analytical decision of the problems (6, 7) is possible only for some special cases of changing of N(z) function. At the smooth changing of the function N(z) the WKBJ approximation method is frequently applied for the approximate calculation of the own values and the own functions. However this approach is limited by the case, when the function N(z) has no more than one maximum.

More accurate results may be received by direct use of the numerical methods, and at the present tome there are several methods of the numerical solution of the problems (6, 7): the finite-difference approximation method, at which the differential equations (6, 7) and the boundary conditions are replaced with the system of the difference equations, approximation of the initial continuous distribution of density of the piecewise-constant function. In this case there is a possibility of existence of only the final number of the wave modes. The analysis of the asymptotic behavior of the phases velocities c_f in the shortwave field has demonstrated, that in the stratified medium with the step-by-step stratification $c_f\approx k^{-1/2}$, while for the medium with the continuous profile of density $c_f\approx k^{-1}$. Piecewise constant approximation of the Vaisala-Brunt frequency. The numerical solution of the differential equations, derived from (6, 7) after introduction of the Prewfer modified transformations

$$\phi(z)=\exp(az)\sin b(z) \quad (8)$$

$$\frac{d\phi(z)}{dz}=\exp(az)\cos b(z)$$

As a result of the transformations (8) for definition of the dispersive dependences it is enough to solve the nonlinear boundary value problem of the first order for the function $b(z)$, behavior of which unlike $\phi(z)$ is monotonous.

The up to now cumulative experience of calculation of dispersive dependences demonstrates, that their most complex behavior arises at the presence in the stratified medium of several wave guides and on the charts of the dispersive curves there may arise the nodes and crowdings, which testify, that the behavior of the group speeds of the internal waves becomes non-monotonic and on some (abnormal) frequencies the different modes extend practically with the identical phase speeds, having the different group speeds. Such areas are called the resonant zones and in them conditions for an overflow of the energy from the lowest energy-carrying modes into the highest energy-carrying modes are created. This phenomenon looks like as insignificant in application to the linearized problem, but may be important at considering the nonlinear members. The abnormal frequencies represent the rather important feature of the internal waves, on them there is a qualitative change of the vertical structure of the wave field.

The thin structure of distribution of the Vaisala-Brunt frequency also may bring to the similar effects of the crowding of the dispersive characteristics on depth. The dispersive curves under action of the thin hydrological structure can be stratified into the separate groups (clusters) inside which occurs the rapprochement of the dispersive parameters of the different mode, whereas the groups themselves are moving away from each other. Such stratification, apparently, may affect on the spectra of the internal waves in the field of the frequencies close to the maximum Vaisala-Brunt frequency.

For the solution of the equations (1) with conditions of (2, 3) rather convenient method of solution is application of Green functions describing development of generations caused by an instant dot source, being on the depth of z_1. In case of the system homogeneous in the horizontal direction it is useful to use Fourier expansion

$$G(\mathbf{r},z,z_1,t)=\int\frac{d\mathbf{k}}{(2\pi)^2}\frac{d\omega}{2\pi}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}G(\mathbf{k},z,z_1,\omega) \qquad (9)$$

Then $G(\mathbf{k},z,z_1,\omega)$ should satisfy the equation of the following kind:

$$LG(\mathbf{k},z,z_1,\omega)=\delta(z-z_1) \qquad (10)$$

$$L=\frac{\partial}{\partial z}\rho_0(z)\frac{\partial}{\partial z}+\rho_0k^2\left(\frac{N^2(z)}{\omega^2}-1\right)$$

The solution of this equation one should look for in the form of the eigenfunctions expansion of the problem (6)

$$G(\mathbf{k},z,z_1,\omega)=G_0(\mathbf{k},z,z_1)+\sum_n\frac{\omega_n^2(k)\phi_n(k,z)\phi_n(k,z_1)}{\omega^2-\omega_n^2(k)}$$

where $G_0(\mathbf{k},z,z_1)$ is the solution of the equation (10) at $\omega\to\infty$ and describes an instant part of the medium response to the external excitation, and the sum of the eigenfunctions describes the contribution of the wave part. Usually the value G_0 is rejected without any discussions. However in some cases, for example, at calculation of the amplitudes of the waves from the periodic sources, this component may be essential, because for the internal waves the law of decrease of the amplitudes of the wave and the non- wave parts of the excitation as the distance from the source of excitation increases is identical.

At fulfillment of the inverse Fourier transformation there is an ambiguity connected with the necessity to set the rule for the flow past the singularities on the real axis ω. The choice of the unambiguous solution is achieved at imposing the causality requirement being reduced to the condition $G(t)\big|_{t<0}\equiv 0$ (Green's retarded function). Green's retarded function corresponds to the solution satisfying the principle of Mandelshtam radiation, when the energy expands from the source. By virtue of the specific law of dispersion of the internal waves the Mandelshtam radiation condition sometimes does not coincide with the Zommerfeld radiation condition (the waves leaving the source), but the use of the Zommerfeld radiation principle at the choice of the unambiguous solution can lead to the incorrect results.

One more method of the choice of the unambiguous solution is the method attributed to Relay providing for introduction of the infinitesimal dissipation equivalent to the Mandelshtam condition. Often the additional condition is set in the form of the requirement of absence of the wave excitations in the distant area upwards the stream (Long condition) however the universality of this condition is not obvious at considering the effect of blocking observable in the stratified liquids. It is also possible to use the approach, at which the stationary solution is considered as a limit at $t\to\infty$ of the non-stationary solution for the acting in the stream source of excitation with the constant characteristics, and which is put into operation $t=0$.

Let us underline, that the causality condition for Green's function is equivalent to the requirement of analyticity of its transient Fourier transformation in the upper half-plane of the complex frequencies ω. It means, that the features on the real axis should be flow past from above, or in accordance with Feynman rule, to exercise the substitution $\omega\to\omega+i\varepsilon\quad(\varepsilon\to+0)$, having shifted the features from the real axis downwards. The analyticity of the transient Fourier transformation in the upper half-plane ω enables to write the Cramers-Cronig ratios expressing relationship between the real and the imaginary parts of Green function, and also in the case of $N(z)=const$ by simple way to construct Green function by means of the analytical continuation from the “non-wave" field of $\omega^2 > N^2$ into the "wave" field of $\omega^2 < N^2$ (the fields, where the equation of the internal waves belongs accordingly to the elliptic or the hyperbolic type).

1.2. Wave Dynamics in Horizontally Inhomogeneous Mediums

As is well known, an essential influence of the propaganda of internal gravity waves in stratified natural mediums (Arctic basin) is caused by the horizontal inhomogeneity and non-stationarity of these media. To the most typical horizontal inhomogeneitities of a real ocean one can refer the modification of the relief of the bottom, and inhomogeneity of the density field, and the variability of the mean flows. One can obtain an exact analytic solution of this problem (for instance, by using the method of separation of variables) only id the distribution of density and the shape of the bottom are described by rather simple model functions. If the shape of the bottom and the stratification are arbitrary, then one can construct only asymptotic representation of the solution in the near and far zones; however, to describe the field of internal waves between these zones, one needs an accurate numerical solution of the problem.

Using asymptotic methods, one can consider a wide class of interesting physical problems, including problems concerning the propagation of non-harmonic wave packets of internal gravity waves in diverse non-homogeneous stratified media under the assumption that the modification of the parameters of a vertically stratified medium are slow in the horizontal direction. From the general point of view, problems of this kind can be studied in the framework of a combination of the adiabatic and semi-classical approximations or by using close approach, for example, ray expansions. In particular, the asymptotic solutions of diverse dynamical problems can be described by using the Maslov canonical operator, which determines the asymptotic behavior of the solution, including the case of neighborhoods of singular sets composed of focal points, caustics, etc. [20, 21]. The specific form of the wave packet can be finally expressed by using some special functions, slay, in terms of oscillating exponentials, Airy function, Fresnel integral, Pearcey-type integral, etc. The above approaches are quite general and, in principle, enable one to solve a broad spectrum of problems from the mathematical point of view; however, the problem of their practical applications and, in particular, of the visualization of the corresponding asymptotic formulas based on the Maslov canonical operator is still far from completion, and in some specific problems to find the asymptotic behavior whose computer realization using software of Mathematica type is rather simple. In this paper, using the approaches developed in [14, 15, 20, 21], we construct and numerically realize asymptotic solutions of the problem, which is formulated as follows.

If we examine the internal gravity waves dynamics for the case when the undisturbed density field $\rho_0(z,x,y)$ depends not only on the depth z, but on the horizontal coordinates x and y, then, in general terms, if the undisturbed density is a function of horizontal coordinates, such a distribution of density induces a field of horizontal flows. These flows, however, are extremely slow and in the first approximation can be neglected. So it is commonly supposed that the field $\rho_0(z,x,y)$ is defined a priori, thus, it is assumed that there exist certain external sources or the examined system is non-conservative. It is also evident that if the internal gravity waves are propagating above an irregular bottom there is no such a problem, because the "internal wave–irregular bottom" system is conservative and there is no external energy flush.

Then we investigated the following liberalized system of equations of hydrodynamics [14, 15, 17]

$$\rho_0 \frac{\partial U_1}{\partial t} = -\frac{\partial p}{\partial x}$$

$$\rho_0 \frac{\partial U_2}{\partial t} = -\frac{\partial p}{\partial y}$$

$$\rho_0 \frac{\partial W}{\partial t} = -\frac{\partial p}{\partial z} + g\rho \tag{11}$$

$$\frac{\partial U_1}{\partial x} + \frac{\partial U_2}{\partial y} + \frac{\partial W}{\partial z} = 0$$

$$\frac{\partial \rho}{\partial t} + U_1 \frac{\partial \rho_0}{\partial x} + U_2 \frac{\partial \rho_0}{\partial y} + W \frac{\partial \rho_0}{\partial z} = 0$$

Here (U_1, U_2, W) is the velocity vector of internal gravity waves, p and ρ are the pressure and density perturbations, g is the acceleration of gravity (z-axis is directed downwards).

Using the Boussinesq approximation which means the density $\rho_0(z,x,y)$ in the first three equations of the system (11) is assumed a constant value, the system (11) by applying the cross-differentiating will be given as

$$\frac{\partial^4 W}{\partial z^2 \partial t^2}+\Delta\frac{\partial^2 W}{\partial t^2}+\frac{g}{\rho_0}\Delta(U_1\frac{\partial\rho_0}{\partial x}+U_2\frac{\partial\rho_0}{\partial y}+W\frac{\partial\rho_0}{\partial z})=0 \quad (12)$$

$$\frac{\partial}{\partial t}(\Delta U_1+\frac{\partial^2 W}{\partial z\partial x})=0\ ,\ \frac{\partial}{\partial t}(\Delta U_2+\frac{\partial^2 W}{\partial z\partial y})=0$$

As the boundary conditions we take the "rigid-lid" condition: $W=0$ at $z=0,H$. Consider the harmonic waves $(U_1,U_2,W)=\exp(i\omega t)(U_1,U_2,W)$. Introduce the non-dimensional variable according to the formulas: $x^*=\frac{x}{L},\ \ y^*=\frac{y}{L},\ \ z^*=\frac{z}{h}$, where L is the typical scale of the horizontal variations ρ_0; h is the typical scale of the vertical variations ρ_0 (for example, the thermocline width).

In non-dimension coordinates the equation system (12) will be written as (index $*$ is omitted hereafter)

$$-\omega^2(\frac{\partial^2 W}{\partial z^2}+\varepsilon^2\Delta W)+$$
$$+\varepsilon^2\frac{g_1}{\rho_0}(\varepsilon U_1\frac{\partial\rho_0}{\partial x}+\varepsilon U_2\frac{\partial\rho_0}{\partial y}+W\frac{\partial\rho_0}{\partial z})=0 \quad (13)$$

$$-\omega^2(\frac{\partial^2 W}{\partial z^2}+\varepsilon^2\Delta W)+$$
$$+\varepsilon^2\frac{g_1}{\rho_0}(\varepsilon U_1\frac{\partial\rho_0}{\partial x}+\varepsilon U_2\frac{\partial\rho_0}{\partial y}+W\frac{\partial\rho_0}{\partial z})=0$$

$$\varepsilon\Delta U_1+\frac{\partial^2 W}{\partial z\partial x}=0\ ,\ \varepsilon\Delta U_2+\frac{\partial^2 W}{\partial z\partial y}=0$$

$$\varepsilon=\frac{h}{L}<<1\ ,\ g_1=\frac{g}{h}\,.$$

The asymptotic solution (14) shall be found in the form usual for the geometric optics method

$$\mathbf{V}(z,x,y)=\sum_{m=0}^{\infty}(i\varepsilon)^m\mathbf{V}_m(z,x,y)\exp(\frac{S(x,y,t)}{i\varepsilon}) \quad (14)$$

$$\mathbf{V}(z,x,y)=(U_1(z,x,y),U_2(z,x,y),W(z,x,y))$$

Functions $S(x,y,t)$ and $\mathbf{V}_m, m=0,1,...$ are subject to definition. From here on we shall restrict ourselves to finding only the dominant member of the expansion (15) for the vertical velocity component $W_0(z,x,y)$, at that from the last two equations (13) we have

$$U_{10}=-\frac{i\partial S/\partial x}{|\nabla S|^2}\frac{\partial W_0}{\partial z}\,,U_{20}=-\frac{i\partial S/\partial y}{|\nabla S|^2}\frac{\partial W_0}{\partial z} \quad (15)$$

$$|\nabla S|=\left(\frac{\partial S}{\partial x}\right)^2+\left(\frac{\partial S}{\partial y}\right)^2$$

Substitute (14) into the first equation of the system (13) and set equal the members of the order *O(1)*

$$\frac{\partial^2 W_0}{\partial z^2}+|\nabla S|^2\left(\frac{N^2(z,x,y)}{\omega^2}-1\right)W_0=0 \quad (16)$$

$$W_0(0,x,y)=W_0(H,x,y)=0$$

where $N^2(z,x,y)=\frac{g_1}{\rho_0}\frac{\partial\rho_0}{\partial z}$ is the Vaisala-Brunt frequency depending of the horizontal coordinates.

The boundary problem (16) has a calculation setup of eigenfunctions W_{0n} and eigenvalues $K_n(x,y)\equiv|\nabla S_n|$, which are assumed to be known. From here on the index n will be omitted while assuming that further calculations belong to an individually taken mode.

For the function $S(x,y)$ we have the eikonal equation

$$\left(\frac{\partial S}{\partial x}\right)^2+\left(\frac{\partial S}{\partial y}\right)^2=K^2(x,y) \quad (17)$$

Initial conditions for the eikonal S for the horizontal case are defined on the line $L: x_0(\alpha),y_0(\alpha)$: $S(x,y)|_L=S_0(\alpha)$. For solving the eikonal equation we construct the rays, that is, the equation (18) with characteristics (rays)

$$\frac{dx}{d\sigma}=\frac{p}{K(x,y)}\quad\frac{dp}{d\sigma}=\frac{\partial K(x,y)}{\partial x} \quad (18)$$

$$\frac{dy}{d\sigma}=\frac{q}{K(x,y)}\quad\frac{dq}{d\sigma}=\frac{\partial K(x,y)}{\partial y}$$

where $p=\partial S/\partial x,\ q=\partial S/\partial y,\ d\sigma$ is the length element of the ray. The initial conditions p_0 and q_0 shall be defined from the system

$$p_0\frac{\partial x_0}{\partial\alpha}+q_0\frac{\partial y_0}{\partial\alpha}=\frac{\partial S_0}{\partial\alpha}$$

$$p_0^2+q_0^2=K^2(x_0(\alpha),y_0(\alpha))$$

The equations (3.1.9) and initial conditions $x_0(\alpha),y_0(\alpha),p_0(\alpha),q_0(\alpha)$ define the ray $x=x(\sigma,\alpha), y=y(\sigma,\alpha)$. After the rays are found the eikonal S is defined by integration along the ray:

$$S = S_0(\alpha) + \int_0^{\sigma} K(x(\sigma,\alpha), y(\sigma,\alpha))d\sigma$$

The function W_0 is defined to the accuracy of multiplication by the arbitrary function $A_0(x,y)$. We shall find W_0 given as: $W_0(z,x,y) = A_0(x,y)W_0^*(z,x,y)$, where $W_0^*(z,x,y)$ is the solution of the vertical spectral problem (17) normalized as follows

$$\int_0^H (N^2(z,x,y) - \omega^2)W_0^{*2}(z,x,y)dz = 1$$

Finally, we can obtain a following equation

$$\nabla A_0^2 \nabla S + A_0^2 \Delta S - 3\nabla S \nabla \ln K = 0$$

This equation will be solved in characteristics of the eikonal equation (18). Using the formula for ΔS along the rays: $\Delta S = \frac{1}{J}\frac{d}{d\sigma}(JK)$, where $J(x,y)$ is the geometric ray spread, we reduce the transfer equation (19) to the following conservation law along the rays

$$\frac{d}{d\sigma}\left(\ln \frac{A_0^2(x,y)J(x,y)}{K^2(x,y)}\right) = 0$$

Note that the wave energy flash is proportional to $A_0^2 K^{-1} da$, thus, from this equation it follows that, in this case, there survives the value equal to the wave energy flash divided by the wave vector modulus.

To proceed to studying the problem of non-harmonic wave packets evolution in a smoothly non-uniform horizontally and non-stationary stratified medium we presuppose the choice of Anzatz ("Anzatz" is the German for a solution type), which define the propagation of Airy and Fresnel internal waves with certain heuristic arguments. The Airy waves describe the features of far wave internal gravity fields in shelf zone, the Fresnel waves describe the features of far wave internal gravity fields in deep ocean [14, 15, 22].

Airy wave. Let's introduce the slow variables $x^* = \varepsilon x,\ \ y^* = \varepsilon y,\ \ t^* = \varepsilon t$ (no slowness is supposed over z, the index is omitted hereafter), where $\varepsilon = \lambda / L << 1$ is the small parameter that characterizes the softness of ambient horizontal changes (λ is the typical iternal gravity wave length, L is the scale of a horizontal non-uniformity). Next we examine the superimposition of harmonic waves (in slow variables x,y,t)

$$W = \int \omega \sum_{m=0}^{\infty} (i\varepsilon)^m W_m(\omega,z,x,y)\ \exp(F)d\omega$$

$$F(x,y,t) = \frac{i}{\varepsilon}\left[\omega t - S_m(\omega,x,y)\right]$$

With respect to functions $S_m(\omega,x,y)$ it is assumed that they are odd-numbered on ω and $\min_{\omega} \partial S / \partial \omega$ is reached at $\omega = 0$ (for all x and y). Substituting this representation into (20) we can easily have it proved that the function $W_m(\omega,z,x,y)$ has at $\omega = 0$ a pole of the m-th order. Therefore, as the model integral $R_m(\sigma)$ for individual terms will serve the following formulas

$$R_m(\sigma) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{i}{\omega}\right)^{m-1} \exp(i\omega^3/3 - i\sigma\omega)d\omega$$

where the integration contour is going around the point $\omega = 0$ from overhead, which enables the functions $R_m(\sigma)$ to exponentially decay at $\sigma >> 1$. The functions $R_m(\sigma)$ have the following feature:

$\frac{dR_m(\sigma)}{d\sigma} = R_{m-1}(\sigma)$, at that $R_0(\sigma) = Ai'(\sigma)$, $R_1(\sigma) = Ai(\sigma)$, $R_2(\sigma) = \int_{-\infty}^{\sigma} Ai(u)du$, etc. It is evident, considering certain properties of Airy integrals, that the functions $R_m(\sigma)$ related with each other as

$$R_{-1}(\sigma) + \sigma R_1(\sigma) = 0$$

$$R_{-3}(\sigma) + 2R_0(\sigma) - \sigma^2 R_1(\sigma) = 0\ .$$

Fresnel wave. As the model integrals $R_m(\sigma)$ that describe the propagation of Fresnel waves taking into account the solution structure for the displacement in the horizontally uniform case we use the following formulas

$$R_0(\sigma) = \mathrm{Re}\int_0^{\infty} \exp\left(-it\sigma - it^2/2\right)dt$$

$$R_{-1}(\sigma) + i\sigma R_0(\sigma) = 0$$

$$R_{-3}(\sigma) - 2iR_{-1}(\sigma) - i\sigma R_{-2}(\sigma) = 0$$

Based on the above, and as well on the first member structure of the Airy and Fresnel uniform wave asymptotics for a horizontally uniform medium, the solution of the system in (20) can be found, for instance, in the form (for an individually taken mode W_n, $\mathbf{U}_n$, further omitting the index n)

$$W = \varepsilon^0 W_0(z,x,y,t)R_0(\sigma) +$$

$$+\varepsilon^a W_1(z,x,y,t)R_1(\sigma)+\varepsilon^{2a}W_2(z,x,y,t)R_2(\sigma)+...$$

$$\mathbf{U}=\varepsilon^{1-a}\mathbf{U}_0(z,x,y,t)R_1(\sigma)+$$
$$+\varepsilon\mathbf{U}_1(z,x,y,t)R_2(\sigma)+\varepsilon^{1+a}\mathbf{U}_2(z,x,y,t)R_3(\sigma)+...$$

where the argument $\sigma=\left(S(x,y,t)/a\right)^a\varepsilon^{-a}$ is assumed to be of the order of unity. This expansion agrees with a common approach of the geometric optics method and space-time ray-path method.

Note also that from such a solution structure it follows that the solution for a horizontally non-uniform and non-stationary medium shall depend on both the "fast" (vertical coordinate) and "slow" (time and horizontal coordinates) variables. Next we generally are going to find a solution in "slow" variables, at that the solution's structural elements which depend on the "fast" variables appear in the form of integrals of some slowly varying functions along the space-time rays.

This solution choice allows us to define the uniform asymptotics for internal gravity wave fields propagating within stratified mediums with slowly varying parameters, which holds true either near or far away from the wave fronts of a single wave mode. If we need only to define the behavior of a field near the wave front, then we can use one of the geometric optics methods – the "progressing wave" method, and a weakly dispersive approximation in the form of appropriate local asymptotics, and find the representation for the phase functions argument σ in the form: $\sigma=\alpha(t,x,y)(S(t,x,y)-\varepsilon t)\varepsilon^{-a}$, where the function $S(t,x,y)$ defines the wave front position and is determined from the eikonal equation solution: $\nabla^2 S=c^{-2}(x,y,t)$, where $c(t,x,y)$ is the maximum group velocity of a respective wave mode, i.e., the first member of the dispersion curve expansion in zero. The function $\alpha(t,x,y)$ (the second member of the expanded dispersion curve) describes the space-time impulse width evolution of Airy or Fresnel non-harmonic internal gravity waves, and then it will be defined from some arbitrary laws of conservation along the eikonal equation characteristics with their actual form to be determined by the problem physical conditions.

2. NUMERICAL SIMULATION

2.1. Wave Dynamics in Vertically Stratified Ocean

Under consideration is the problem of mathematical modeling for the field of steady-state internal gravity waves generated by a non-local disturbing source (for example underwater sea platform) within a flow of stratified medium of the thickness H with an arbitrary distribution of the Vaisala-Brunt frequency $N(z)$. The free surface at $z=0$ is substituted with the "rigid-lid" which allows us to filter off the surface waves, and has little effect upon the internal gravity waves. It is assumed that a flow velocity V exceeds the maximum group velocity of internal waves in real ocean. The disturbing non-local source vertical dimension is considered small as compared to the medium layer thickness. These assumptions mean that the internal Froude number is much greater than unity, so the pictures of the trajectories near a flowing source must qualitatively appear the same as in the case of the uniform (non-stratified) medium [14, 15, 23].

Parameters of the calculations are typical for Arctic basin and underwater sea construction: $N(z)\approx 0.01s^{-1}$, sea depth $H\approx 100m$, stratified flow velocity $V\approx 2m/s$, $\xi=x+Vt$, horizontal scale of underwater streamlined obstacle is about 50 m. The numerical simulation of the problem stated requires quite a number of integrations from the fast oscillating functions, thus; first, we have to use methods which allow us to effectively realize the integrations of this type. Second, the complete wave field near a non-local flowing source of perturbations represents a poorly convergent series, and to obtain an adequate accuracy we have to integrate a large number of modes, however, the use of the static feature discrimination method enables the calculation of the field near a flowing source while avoiding such integrations. Finally, third, at long distances from the source when the complete field falls into singular modes, the asymptotic representations for a single mode of the Green's function, make it possible to calculate the far fields of internal gravity waves without performing exact numerical calculations.

The numerical calculation show, for example, that vertical velocity W is quickly decreasing with decreasing depth and at $z=Z$ ($z=Z$ - depth of the thermocline maximum) it takes about 15% of the velocity value at the bottom. Figure **2** demonstrate the calculation results in the thermocline maximum (integrated were 25 wave modes, the higher modes had not contributed much to the complete field), the maximum value of the displacement at this horizon reached 1.3 meters. The presented results show that there are at least three different regions of the generated field of internal waves. First, it is the region immediately under the non-local source, which has a width of about the medium thickness, it's the near-field region. The numerical calculations have proven that the wave field of internal gravity waves within the near-field region is little dependent on a specific stratification and the velocity amplitude and displacement within this region are maximal. Secondly, at long distances from the non-local source $(y,x>10H,$ the far-field region) the field of internal gravity waves falls apart into singular wave modes, at that each of the modes is contained inside its Mach cone, and outside the cline the amplitude is low. In addition to that there is a transition region in which the structure of the wave fields is rather complex.

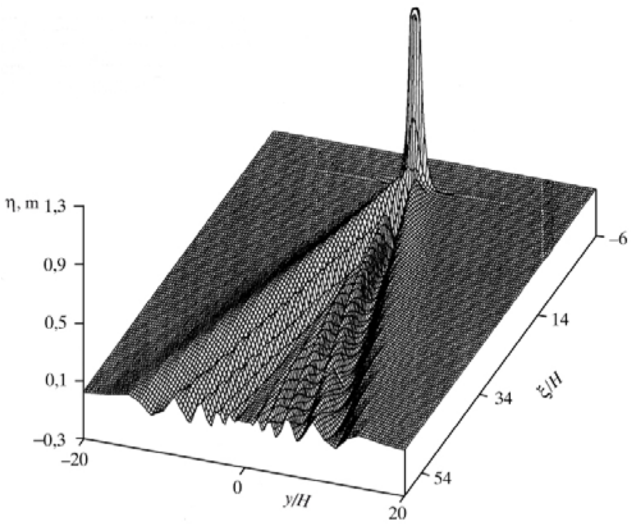

Figure 2: Internal gravity waves (vertical displacement) from a underwater nol-local source in stratified ocean of uniform depth ($\xi = (x + Vt) / H, y = y / H$ - non-dimensional horizontal coordinates).

2.2. Wave Dynamics in Horizontally Inhomogeneous Mediums

In Figure **3** we represent vertical component of internal gravity wave field velocity w generated by a non-local source (underwater obstacle – sea platform) in arbitrary stratified ocean of non-uniform depth. Parameters of the calculations are typical for real ocean (Arctic basin): $N(z) \approx 0.001 s^{-1}$, the slope of the bottom no more than 10^0. Numerical calculations show a significant deformation of the wave field structure, taking into account the horizontal inhomogeneities stratified mediums. For example, it follows from the numerical results thus presented that, outside the caustic, the wave field is sufficiently small indeed and is not subjected to great many oscillations, whereas the wave picture inside the zone of caustic is a rather complicated system of incident and reflected harmonics. It is well known, that caustic is an envelope of a family of rays, and asymptotic solution is obtained along these rays. Asymptotic representation of the field describe qualitative change of the wave field, and that is description of the field, when we cross the area of "light", where wave field exists, and come in the area of "shadow", where we consider wave field to be rather small. Each point of the caustic corresponds to a specified ray, and that ray is tangent at this point. In this paper the most difficult question is considered that can appear when we investigate the problems of wave theory with the help of geometrical optics methods and its modifications. And the main question consists in finding of asymptotic solution near special curve (or surface), which is called caustic [20, 21].

It is a general rule that caustic of a family of rays single out an area in space, so that rays of that family cannot appear in the marked area. There is also another area, and each point of that area has two rays that pass through this point. One of those rays has already passed this point, and another is going to pass the point. Formal approximation of geometrical optics or WKBJ approximation cannot be applied near the caustic, that is because rays merge together in that area, after they were reflected by caustic. If we want to find wave field near the caustic, then it is necessary to use special approximation of the solution, and in the paper a modified ray method is proposed in order to build uniform asymptotic expansion of integral forms of the internal gravity wave field. After the rays are reflected by the caustic, there appears a phase shift. It is clear that the phase shift can only happen in the area where methods of geometrical optics, which were used in previous sections, can't be applied. If the rays touch the caustic several times, then additional phase shifts will be added. Phase shift, which was created by the caustic, is rather small in comparison with the change in phase along the ray, but this shift can considerably affect interference pattern of the wave field.

The asymptotic representations constructed in this paper allow one to describe the far field of the internal gravity waves generated by a non-local sources in stratified flow. The obtained asymptotic expressions for the solution are uniform and reproduce fairly well the essential features of wave fields near caustic surfaces and wave fronts. In this paper the problem of reconstructing non-harmonic wave packets of internal gravity waves generated by a source moving in a horizontally stratified medium is considered. The solution is proposed in terms of modes, propagating independently in the adiabatic approximation, and described as a non-integer power series of a small parameter characterizing the stratified medium. In this study we analyze the evolution of non-harmonic wave packets of internal gravity waves generated by a moving source under the assumption that the parameters of a vertically stratified medium (e.g. an ocean) vary slowly in the horizontal direction, as compared to the characteristic length of the density. A specific form of the wave packets, which can be parameterized in terms of model functions, e.g. Airy functions, depends on local behavior of the dispersion curves of individual modes in the vicinity of the corresponding critical points.

In this paper a modified space-time ray method is proposed, which belongs to the class of geometrical optics methods (WKBJ method) [17, 20, 21]. The key point of the proposed technique is the possibility to derive the asymptotic representation of the solution in terms of a non-integer power series of the small parameter $\varepsilon = \lambda / L$, where λ is the characteristic wave length, and L is the characteristic scale of the horizontal heterogeneity. The explicit form of the asymptotic solution was determined based on the principles of locality and asymptotic behavior of the solution in the case of a stationary and horizontally homogeneous medium. The wave packet amplitudes are determined from the energy conservation laws along the characteristic curves. A typical assumption made in studies on the internal wave evolution in stratified media is that the wave packets are locally harmonic. A modification of the geometrical optics method, based on an expansion of the solution in model functions, allows one to describe the wave field structure both far from and at the vicinity of the wave front.

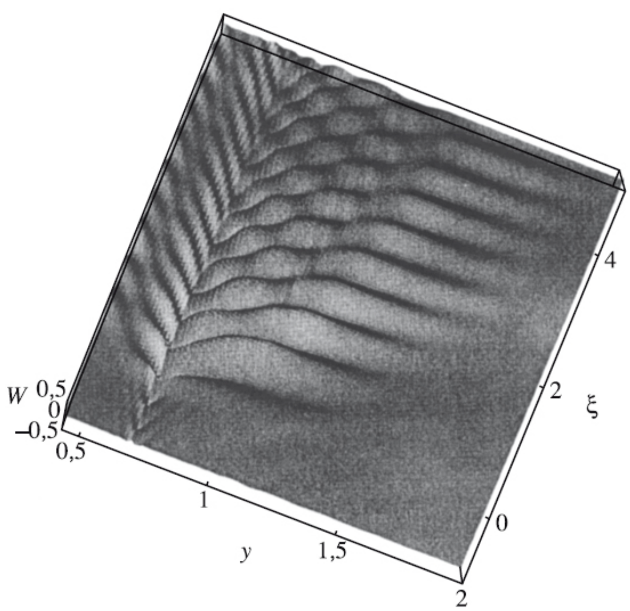

Figure 3: Internal gravity waves (vertical velocity component) from a underwater nol-local source in stratified medium of variable depth ($\xi = (x + Vt)/H, y = y/H$ - non-dimensional horizontal coordinates).

Using the asymptotic representation of the wave field at a large distance from a non-local source in a layer of constant depth, we solve the problem of constructing the uniform asymptotics of the internal waves in a medium of varying depth. The solution is obtained by modifying the previously proposed "vertical modes-horizontal rays" method, which avoids the assumption that the medium parameters vary slowly in the vertical direction. The solution is parameterized, for example, through the Airy waves. This allows one to describe not only the evolution of the non-harmonic wave packets propagating over a slow-varying fluid bottom, but also specify the wave field structure associated with an individual mode both far from and close to the wave front of the mode. The Airy function argument is determined by solving the corresponding eikonal equations and finding vertical spectra of the internal gravity waves. The wave field amplitude is determined using the energy conservation law, or another adiabatic invariant, characterizing wave propagation along the characteristic curves [14, 15, 22].

Modeling typical shapes and stratification of the ocean shelf we obtain analytic expressions describing the characteristic curves and examine characteristic properties of the wave field phase structure. As a result it is possible to observe some peculiarities in the wave field structure, depending on the shape of ocean bottom, water stratification and the trajectory of a moving source. In particular, we analyze a spatial blocking effect of the low-frequency components of the wave field, generated by a source moving alongshore with a supercritical velocity. Numerical analyses that are performed using typical ocean parameters reveal that actual dynamics of the internal gravity waves are strongly influenced by horizontal non-homogeneity of the ocean bottom. In this paper we use an analytical approach, which avoids the numerical calculation widely used in analysis of internal gravity wave dynamics in stratified ocean.

CONCLUSIONS

The main fundamental problems of wave dynamics considered in the present paper were the following:

- construction of the exact and asymptotic solutions of the problem concerning the internal gravity waves excited by the non-local disturbing sources in the non-uniform stratified mediums, as well as development of the numerical algorithms for analysis of the corresponding spectral problems and for calculation of the wave disturbances for the real parameters of the vertically stratified mediums;
- research by means of the modified version of the space-time ray-tracing method (WKBJ method), evolution of the non-harmonic wave-trains of the

internal gravity waves in the supposition of the slowness of variation of the parameters of the vertically stratified medium in the horizontal direction and in a time;

- the asymptotic analysis of the critical modes of generation and propagation of the internal gravity waves in the stratified mediums, including the study of the effects of the space-frequency screening;

- development of non-spectral methods of analysis of the in-situ measurements of the internal gravity waves for the purpose of the possible distant definition of the characteristics of the broad-band wave-trains, composing the measured hydrophysical fields, as well as the parameters of the ocean along a line of propagation of these wave-trains.

The paper presented methods and approaches of research of the internal gravity waves dynamics combine the comparative simplicity and computational capability to gain the analytical results, the possibility of their qualitative analysis and the accuracy of the numerical results. Besides that there is a possibility of inspection of the trustworthiness of the used hypotheses and approximations on the basis of analysis of the real oceanological data, while the exact analytical solutions for the model problems do not allow to apply the gained outcomes, for example, for analysis of the problem with the real parameters of the medium, and the exact numeric calculation for one particular real medium does not give the possibility of the qualitative analysis of the medium with other real parameters.

The results presented by the paper on the research of the dynamics of the non-harmonic wave-trains of the internal waves in the stratified mediums with the varying parameters enable analytically and numerically to examine effects of the special blocking, and also the excitation and failure of the separate frequency components of the propagating wave-trains.

It is necessary to mark once again, that in comparison with the majority of the researches devoted to study of the dynamics of the internal gravity waves, the methods of decomposing of the fields of the internal gravity waves into the certain benchmark functions enable to describe the main peculiarities of formation of the critical modes of generation and propagation of the non-harmonic wave-trains. It is expedient also to emphasize, that the built asymptotic representations in the form of the applicable model functions can be used also for study of any other wave processes (acoustical and seismic waves, SHF-irradiation, the tsunami waves, etc.) in the real mediums with a complex structure. All fundamental results of the paper are gained for the arbitrary distributions of the density and other parameters of the non-uniform media, and besides the main physical mechanisms of formation of the studied phenomena of the dynamics of the internal gravity waves in the non-uniform stratified mediums were considered in the context of the available data of the in-situ measurements.

The next step in the asymptotic study of the internal gravity waves should be study of the linear interaction of the wave-trains at their propagation as we used approximation of adiabatic, that is the independence of wave modes from each other. However, generally, the linear interaction (the linear conversion) of the waver modes is present. The phenomenon of the linear conversion of the internal gravity waves consists, that at the wave-trains passing through the non-uniform sections of the medium the amplitudes of the waves can vary non-adiabatically, that is the real amplitude-phase characteristics of the fields are varying differently, than it follows from the fundamental approximations of the geometrical optics used in this paper. The detailed study of these problems will be the subject of further researches.

The universal nature of the he asymptotic methods of research of the internal gravity waves offered in this paper is added with the universal heuristic requirements of the applicability of these methods. These criteria ensure the internal control of applicability of the used methods, and in some cases on the basis of the formulated criteria it is possible to evaluate the wave fields in the place, where the given methods are inapplicable. Thus there are the wide opportunities of analysis of the wave patterns as a whole, that is relevant both for the correct formulation of the analytical investigations, and for realization of estimate calculations at the in-situ measurements of the wave fields.

The special role of the given methods is caused by that condition, that the parameters of the natural stratified mediums, as a rule, are known approximately, and efforts of the exact numerical solution of initial equations with usage of such parameters can lead to the overstatement of accuracy.

Also popularity of the used approaches of analysis of internal gravity wave dynamics can be promoted just by the existence of the lot of the interesting physical problems quite adequately described by these approaches and can promote the interest to the multiplicity of problems bound to a diversification of the non-uniform stratified mediums. The value of such methods of analysis of the wave fields is determined not only by their obviousness, scalability and effectiveness at the solution of the different problems, but also that they can be some semi-empirical basis for other approximate methods in theory of propagation of the internal gravity waves.

The results of this paper represent significant interest for physics and mathematics. Besides, asymptotic solutions, which are obtained in this paper, can be of significant importance for engineering applications, since the method of geometrical optics, which we modified in order to calculate the wave field near caustic, makes it possible to describe different wave fields in a rather wide class of other problems

ACKNOWLEDGEMENTS

The results presented in the paper have been obtained by research performed under projects supported by the Russian Foundation for Basic Research (No.11-01-00335a, No. 13-05-00151a), Program of the Russian Academy of Sciences "Fundamental Problems of Oceanology: Physics, Geology, Biology, Ecology".

REFERENCES

[1] Nansen F. Farthest North: The epic adventure of a visionary explorer. Skyhorse Publishing 1897.

[2] Ekman VW. On dead water. In: Nansen F, editor. The Norwegian north polar expedition 1893-1896. Scientific Results. Kristiania 1904; pp. 152-166.

[3] Apel JR. Observations of internal wave surface signatures in ASTP photographs. In: Apollo-Soyuz Test Project. Summary Science Report. NASA SP. Washington: Scientific and Technical Information 1978; Vol. II: pp. 505-509.

[4] Dickey JO, *et al.* Luna laser ranging: A continuing legacy of the Apollo Program. Science 1994; 265: 482-90. http://dx.doi.org/10.1126/science.265.5171.482

[5] Morozov EG. Internal tides. Global field of internal tides and mixing caused by internal tides. In: Grue J, Trulsen K, Eds. Waves in geophysical fluids. Wein New York: Springer 2006; pp. 271-32. http://dx.doi.org/10.1007/978-3-211-69356-8_6

[6] Grue J. Very large internal waves in the ocean – observations and nonlinear models. In: Grue J, Trulsen K, Eds. Waves in geophysical fluids. Wein New York: Springer 2006; pp. 205-207. http://dx.doi.org/10.1007/978-3-211-69356-8_5

[7] Grue J, Sveen JK. A scaling law of internal run-up duration. Ocean Dynamics 2010; 60: 993-1006. http://dx.doi.org/10.1007/s10236-010-0284-4

[8] Alendal G, Berntsen J, Engum E, Furnes GK, Kleiven G, Eide LI. Influence from 'Ocean Weather' on near seabed currents and events at Ormen Lange. Marine Petroleum Geology 2005; 22: 21-31. http://dx.doi.org/10.1016/j.marpetgeo.2004.10.011

[9] Garret C. Internal tides and ocean mixing. Science 2003; 301: 1858-59.

[10] Garrett C, Kunze E. Internal tide generation in the deep ocean. Rev Fluid Mech 2007; 39: 57-87. http://dx.doi.org/10.1146/annurev.fluid.39.050905.110227

[11] Grue J. Internal wave fields analyzed by imaging velocimetry. In: Grue J, Liu PLF, Pedersen GK, Eds. PIV and Water Waves. World Scientific 2004; pp. 239-278. http://dx.doi.org/10.1142/9789812796615_0006

[12] Song ZJ, Gou BY, Lua L, Shi ZM, Xiao Y, Qu Y. Comparisons of internal solitary wave and surface wave actions on marine structures and their responses. Appl Ocean Res 2011; 33: 120-29. http://dx.doi.org/10.1016/j.apor.2011.01.003

[13] Hsu MK, Liu AK, Liu C. A study of internal waves in the China Seas and Yellow Sea using SAR. Continental Shelf Res 2000; 20: 389-10. http://dx.doi.org/10.1016/S0278-4343(99)00078-3

[14] Bulatov VV, Vladimirov YuV. Internal gravity waves: theory and applications. Moscow: Nauka Publishers 2007.

[15] Bulatov VV, Vladimirov YuV. Wave dynamics of stratified mediums. Moscow: Nauka Publishers 2012.

[16] Miropol'skii YuZ, Shishkina OV. Dynamics of internal gravity waves in the ocean. Boston: Kluwer Academic Publishers 2001.

[17] Pedlosky J. Waves in the ocean and atmosphere: introduction to wave dynamics. Berlin- Heidelberg: Springer 2010.

[18] Sutherland BR. Internal gravity waves. Cambridge: Cambridge University Press 2010.

[19] Lighthhill MJ. An informal introduction to theoretical fluid mechanics. Oxford: Oxford University Press 1986.

[20] Babich VM, Buldyrev VS. Asymptotic methods in short-wavelenght diffraction theory. Oxford: Alpha Science 2007.

[21] Arnold VI. Catastrophe theory. Berlin, Heidelberg: Springer 1992.

[22] Bulatov VV, Vladimirov YuV. The uniform asymptotic form of the internal gravity wave field generated by a source moving above a smoothly varying bottom. J Eng Math 2011; 69(Pt 2): 243-60. http://dx.doi.org/10.1007/s10665-010-9388-6

[23] Voizin B. Limit states of internal wave beams. J Fluid Mech 2003; 496: 243-93. http://dx.doi.org/10.1017/S0022112003006414

Received on 13-12-2012 Accepted on 24-01-2013 Published on 06-02-2013

http://dx.doi.org/10.6000/1927-5129.2013.09.12