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# Far Fields of Internal Gravity Waves in a Stratified Liquid of Varying Depth

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**Abstract**—Asymptotic representations of solutions describing the far fields of internal gravity waves in a stratified medium of varying depth are constructed. The effect of space-frequency cutoff of the wave field for a real oceanic shelf is revealed. Depending on frequency characteristics of the wave field and bottom topography, far fields of internal waves either are located in a certain confined space domain (trapped waves) or propagate in the absence of turning points over sufficiently large distances when compared with the sea depth (progressive waves). The space domain where the progressive waves penetrate is fully determined by the presence of turning points whose locations depend on the medium stratification and inhomogeneities of bottom topography.

*Keywords:* internal gravity waves, stratified media, method of geometrical optics

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## INTRODUCTION

The propagation of internal gravity waves in the ocean is significantly affected by both inhomogeneities of hydrophysical fields (in particular, the density field) and changes in the bottom topography. Moreover, exact analytical solutions to wave problems are derived only if the water-density distribution and the bottom shape are described by sufficiently simple model functions. When the medium characteristics and boundaries are arbitrary, only numerical solutions to these problems can be constructed. However, this circumstance does not allow the wave-field characteristics to be analyzed qualitatively, especially at great distances, which is necessary for solving, e.g., the problem of internal wave detection using remote methods, including aerospace radar aids. In this case wave dynamics can be described and analyzed on the basis of asymptotic models and analytical methods to solve them by means of the modified method of geometrical optics.

## PROBLEM STATEMENT AND SOLVING

We consider a problem regarding far fields of internal gravity waves propagating in a stratified medium of finite depth with a varying bottom relief. In the linear approximation, a system of equations describing small oscillations of the initially resting stratified medium (the $z$ axis is directed vertically downward) has the form [1–3]

$$\mathrm{div}\mathbf{U} = 0,$$
$$\rho_0 \frac{\partial \mathbf{U}}{\partial t} + \mathrm{grad} p + \mathbf{G} = 0, \qquad (1)$$
$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho_0}{\partial z} w = 0,$$

where $\mathbf{U} = (u_1, u_2, w)$, $p, \rho$ are the perturbations of the velocity vector, the pressure, and the density; $\rho_0(z)$ is the medium density in the unperturbed state; and $\mathbf{G} = (0, 0, g\rho)$, $g$ is the acceleration of gravity. A layer of the stratified medium is considered in the Boussinesq approximation with the parameter $N^2(z) = -gd\ln\rho_0/dz = \mathrm{const}$, which is confined with a "rigid lid" ($z = 0$) from above ($w = 0$) and with the bottom $z = -H(y)$, where the impermeability condition $w + \frac{dH}{dy}u_2 = 0$ is satisfied.

Further we introduce the characteristic vertical scale $M$, then $N^2M/g \equiv \lambda^{-1} \ll 1$,; in a real ocean, $\lambda^{-1} \sim 10^{-2}$–$10^{-3}$. Indeed, if the Brunt–Väisälä frequency and bottom depth are constant, we have $\lambda^{-1} = N^2H^2/gH = \pi^2c_{\max}^2/c^2$, where $c = \sqrt{gH}$ is the velocity of long surface waves, while $c_{\max} = NH/\pi$ is the maximum group velocity of internal waves. In a real ocean, $c \sim 100$ m/s, $c_{\max} \sim 1$ m/s [1–3]. We suggest that the function $H(y)$ is a continuously differentiable and smoothly varying function having no more than a single minimum. The smoothness of the $H(y)$ variation implies that the ratio of the horizontal scale $L$ of the $H(y)$ variation to its vertical scale $M$ is characterized by the quantity $\lambda = L/M \gg 1$, which actually means that the bottom has a small inclination. Under the same conditions, the inhomogeneity of bottom topography can be simulated with a single hill, i.e., the $H(y)$ function can be presented with a single maximum. These approximations (the constant stratification $N \approx 10^{-2}$–$10^{-3}$ s$^{-1}$, the bottom relief with a slope no more than 10°) may be observed in certain regions of the World Ocean [4, 5].

A solution to (1) for vertical velocity is sought as $w = \exp(-i\omega t + ilx)W(z, y)$, where $\omega$ is the frequency and $l$ is the horizontal wavenumber. Solutions of a more general form, due to the problem linearity, are obtained by the superposition of the derived asymptotic representations [1, 6, 7]. In the dimensionless variables $x^R = x/L$, $y^R = y/L$, $z^R = z/M$, $l^R = lM$, $\omega^R = \omega/N$, $h(y) = H(Ly)/M$ (index "$R$" is omitted later), we have

$$\frac{\partial^2 W}{\partial z^2} - \frac{1}{\lambda^2 b^2}\frac{\partial^2 W}{\partial y^2} + \frac{l^2}{b^2}W = 0,$$

$$W = 0, \quad \text{at} \quad z = 0, \tag{2}$$

$$W + \frac{dh(y)}{\lambda dy}u_2 = 0, \quad \text{at} \quad z = -h(y), \quad b^2 = \frac{\omega^2}{1-\omega^2}.$$

If the bottom relief is linear ($h(y) = -\gamma y$, $\gamma = 1/\lambda$ is the bottom slope), problem (2) in the zero approximation (i.e., the boundary condition on the bottom acquires the form $W = 0$ at $z = -h(y)$) has an analytical solution [8]:

$$W = \sum W_n, \quad W_n = \exp\left(-i\frac{\pi\eta}{2}\right)$$
$$\times K_\eta\left(l\sqrt{\lambda^2 y^2 - \frac{z^2}{b^2}}\right)\sin\left(\frac{n\pi}{\ln\Delta}\ln\frac{\lambda by - z}{\lambda by + z}\right),$$

where $\Delta = \frac{\lambda b + 1}{\lambda b - 1}$, $\eta = \frac{2\pi ni}{\ln\Delta}$, $K_\eta$ is the Macdonald function of the imaginary index $\eta$.

A solution to (2) for the nonlinear bottom profile is sought in the form typical for the geometrical optics method [6, 7]:

$$W = \left(F_0(z, y, \omega) + \frac{i}{\lambda}F_1(z, y, \omega) + \left(\frac{i}{\lambda}\right)^2 F_2(z, y, \omega) + \ldots\right)$$
$$\times \exp(i\lambda S(y, \omega)),$$

where the functions $F_0$, $F_1$ are defined from the relations

$$\frac{\partial^2 F_0}{\partial z^2} + \left(\left(\frac{\partial S}{\partial y}\right)^2 + l^2\right)F_0/b^2 = 0,$$
$$F_0 = 0 \quad \text{at} \quad z = 0, \ -h(y),$$
$$\frac{\partial^2 F_1}{\partial z^2} + \left(\left(\frac{\partial S}{\partial y}\right)^2 + l^2\right)F_1/b^2 = \frac{1}{b^2}\left(2\frac{\partial F_0}{\partial y}\frac{\partial S}{\partial y} + F_0\frac{\partial^2 S}{\partial y^2}\right), \tag{3}$$
$$F_1 = 0 \quad \text{at} \quad z = 0, \ -h(y).$$

Solving of the first equation from (3) yields the mode structure of the wave with the dispersion relationship $\kappa_n^2(y, \omega) = \frac{b^2 n^2 \pi^2}{h^2(y)}$, $n = 1, 2, \ldots$, and the eigenfunctions in the zero approximation (vertical modes)

$$F_{0n}(z, y, \omega) = A_{0n}(y, \omega)\sin\frac{n\pi z}{h(y)}, \quad n = 1, 2, \ldots .$$

Then the eikonal $S_n(y, \omega)$ is determined from the relation

$$\kappa_n^2(y, \omega) = \left(\frac{\partial S_n}{\partial y}\right)^2 + l^2.$$

The amplitude $A_{0n}(y, \omega)$ is found from the condition of solvability of the second equation from (3), which demands the orthogonality of the right-hand part of this equation and the function $F_{0n}$:

$$A_{0n} = \frac{B_{0n}(y_0, \omega)}{\sqrt[4]{b^2 n^2 \pi^2 - h^2(y) l^2}},$$

where the constant $B_{0n}$ depends on $\omega$ and initial eikonal value at the point $y_0$ ($S_n(y_0, \omega)$. The eikonal $S_n(y, \omega)$ is defined by the expression

$$S_n(y, \omega) = \int_y^{y^*}\sqrt{\kappa_n^2(y, \omega) - l^2}\,dy,$$

where $y^*$ is the turning point, i.e., the root of the equation $\kappa_n^2(y, \omega) = l^2$.

The locus of turning points determines a caustic, in the vicinity of which a qualitative change in properties of wave fields occurs, namely, the transfer from the "light" region, i.e., the domain of existence of wave fields, to a "shadow" region, where these fields are exponentially small. As is known, the caustic from the geometrical viewpoint is the envelope of a family of rays or characteristics along which the asymptotic solution is constructed. The wave-field asymptotics near the caustic describe the qualitative change in properties of the wave fields under investigation. Formal approximations of geometrical optics as well as their modifications become nonapplicable near caustic surfaces. To find a wave field close to the caustic, it

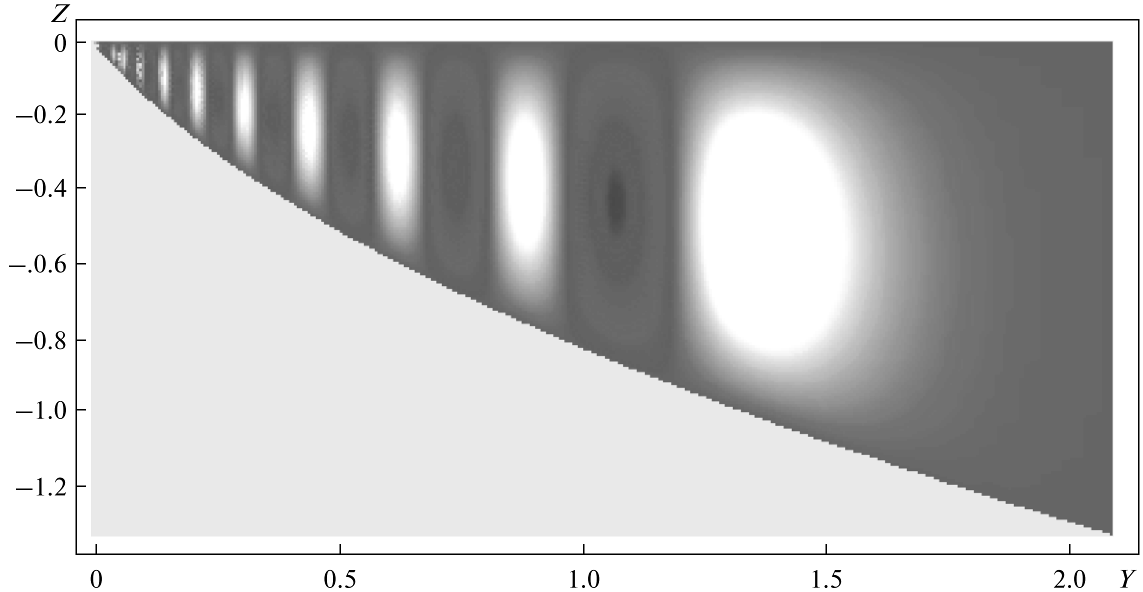


**Fig. 1.** First mode of vertical velocity above the descending bottom profile.

is necessary to build special decompositions of the solution and, e.g., to use the etalon integral method for the construction of the uniform asymptotics of the integral representation of a wave field. Apart from the great physical interest, these asymptotic constructions may be of significant value for applications because the geometrical optics method, modified taking into account a field on the caustic, allows wave fields to be described in a very broad class of problems [1, 6].

Then a solution in the geometrical-optics approximation for an individual wave mode of the far field of internal gravity waves before the turning point, i.e., in the wave region, has the form

$$W_n = \sqrt{2\pi}Q_n\cos\left(\lambda S_n(y,\omega) - \frac{\pi}{4}\right), \quad (4)$$

$$Q_n = \frac{\sin(n\pi z/h(y))}{\lambda^{1/2}\sqrt[4]{b^2n^2\pi^2 - h^2(y)l^2}}.$$

Behind the turning point (in the region of exponential attenuation), this solution can be presented as

$$W_n = \sqrt{\pi}Q_n\exp(-\lambda|S_n(y,\omega)|). \quad (5)$$

The uniform asymptotic of the solution, which can be applied in the vicinity of the turning point, is presented by the expression

$$W_n = \sqrt{2}\pi\left(\frac{3}{2}\lambda S_n(y,\omega)\right)^{1/6}Q_nAi\left(\left(\frac{3}{2}\lambda S_n(y,\omega)\right)^{2/3}\right), \quad (6)$$

where $Ai(x)$ is the Airy function. Solutions (6) at large values of the Airy function argument $\lambda S_n(y,\omega)$ (far from the caustic) coincide with solutions (4) and (5), before and behind the turning point, respectively. Thus, solutions (6) describe most generally a far field of internal gravity waves in a stratified medium of variable depth.

## DISCUSSION OF RESULTS

Figures 1–4 display the results of calculations of the vertical velocity for two typical profiles of the ocean floor, which are distinguished from the linear profile [4, 5]. In Figs. 1 and 2, the results of a calculation of the equipotential lines are presented for the first $W_1(z,y)$ and second $W_2(z,y)$ wave modes, respectively, for the model of the slowly descending profile of the bottom with the value $\omega = 0.55$. In this case $h(0) = 0$, for any $\omega$ with the given wavenumber $l$, turning point $y^*$ exists and trapped waves alone are present. In Figs. 3 and 4, the results of a calculation of the equipotential line are shown for the second wave mode $W_2(z,y)$ for the bottom profile in the form of a smooth hill. In this case $h(0) = h_0 \neq 0$, $h(\infty) = h_\infty$. Then the cut-off frequency $\omega_0 = \Omega(h(0))$, $\Omega(h) = \dfrac{lh}{\sqrt{n^2\pi^2 + l^2h^2}}$ exists, so that waves with the frequencies $\omega < \omega_0$ cannot exist. With $\omega_0 < \omega < \omega_*$ $(\omega_* = \Omega(h_\infty))$, there is a discrete spectrum where, for each frequency $\omega_n$, there is a trapped wave. The frequencies $\omega$ belonging to discrete sets are found using the shooting method while numerically solving the equations $\left.\dfrac{\partial W_n(z,y)}{\partial y}\right|_{y=0} = 0$ for a fixed $z$; $n$ is the mode number [1]. The results of calculations of trapped waves of the second wave mode $W_2(z,y)$ are depicted in Fig. 3 for the values $\omega_0 = 0.238$,

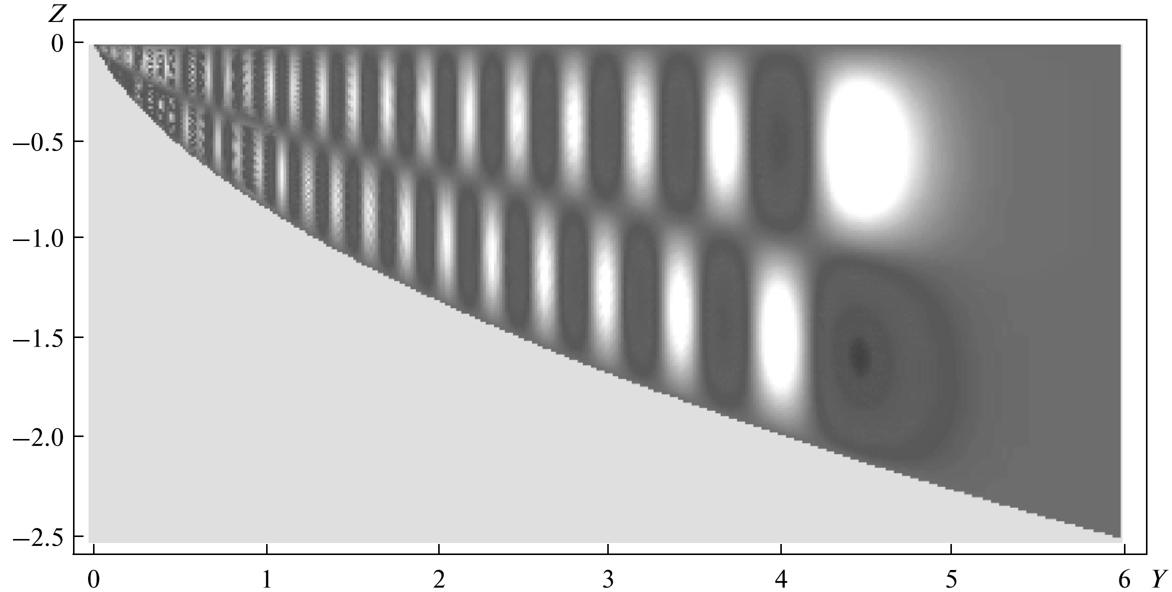


**Fig. 2.** Second mode of vertical velocity above the descending bottom profile.

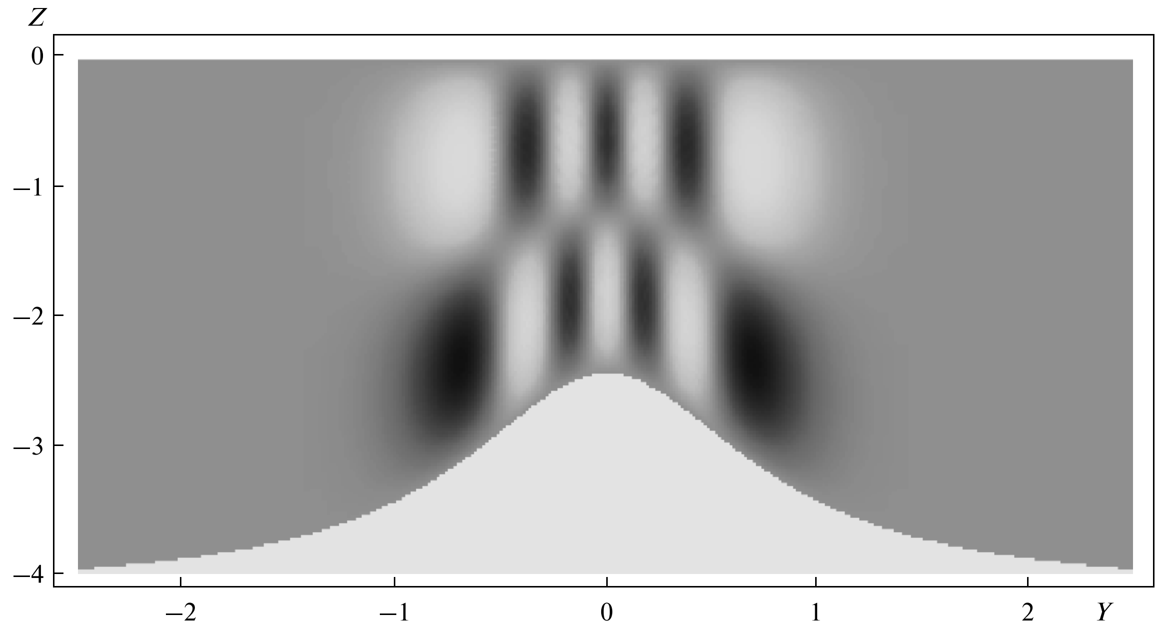


**Fig. 3.** Second mode of vertical velocity above the single hill (trapped waves).

$\omega_* = 0.384$, $\omega = 0.312$. With $\omega_* < \omega < 1$, there are no turning points, the spectrum relative to $\omega$ is continuous, and there are progressive waves; the results of calculations of the second wave mode $W_2(z, y)$ are presented in Fig. 4 for the value $\omega = 0.4$.

Thus, it is shown that different peculiarities may manifest themselves in parameters of far fields of internal waves depending on the bottom shape and structure of the seawater stratification. The effect of a space-frequency "cutoff" of the wave field is revealed for the real oceanic shelf. Depending on frequency characteristics of the wave field and the bottom topography, far fields of internal waves are either localized in a certain confined space domain (trapped waves) or propagate in the absence of turning points over sufficiently large distances when compared with the sea depth (progressive waves). The space domain, where progressive waves penetrate, is completely determined by the presence of turning points whose locations

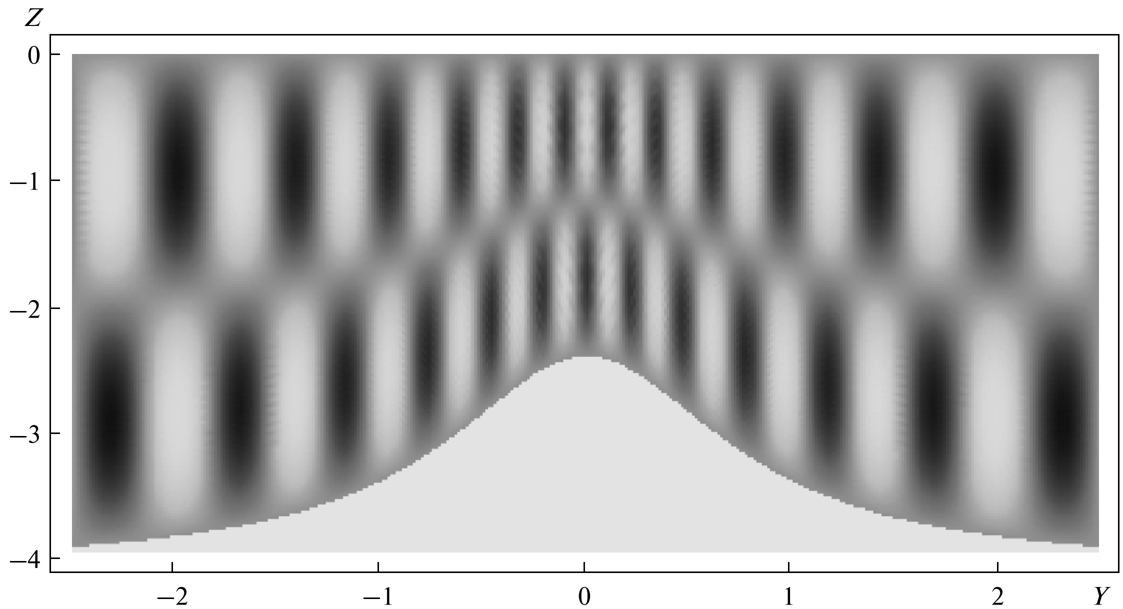


**Fig. 4.** Second mode of vertical velocity above the single hill (progressive waves).

depend on the medium stratifications and irregularities of the bottom topography.

## CONCLUSIONS

The constructed asymptotic solutions are uniform and allow the far fields of internal waves to be described both near and far from turning points. The versatile character of the offered asymptotic method for simulating far fields of internal waves allows one to effectively calculate the wave fields and additionally qualitatively analyze the solutions. Thereby opportunities are created for an analysis of wave patterns as a whole, which is important both for correct statement of mathematical models of wave dynamics and for performing express estimates during the real measurements of wave fields in sea water, including the situation when the ocean stratification is poorly known.

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