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Abstract. In paper fundamental problems of wave dynamic stratified medium modeling (ocean, atmosphere) are considered. The basic mathematical models describing the processes of excitation, the propagation of internal gravity waves in vertical stratified horizontally inhomogeneous and non-stationary medium are presented. The uniform asymptotic forms of the internal gravity waves in horizontally inhomogeneous and non-stationary stratified medium are obtained. A modified spatio-temporal ray method is proposed, which belongs to the class of geometrical optics methods (WKBJ method). Analytical and numerical algorithms of internal gravity wave calculations for the real stratified medium parameters are presented. Some results of internal gravity waves measurements in real stratified medium (ocean, atmosphere) and it's interpretations are discussed. The universal nature of the developed asymptotic methods of modeling makes it possible to efficiently calculate the wave fields and qualitatively analyze the solutions obtained, which makes it possible to analyze wave patterns in general, to correctly set up mathematical models of wave dynamics, and to carry out express estimates in actual measurements of wave fields in natural stratified medium.

1. Introduction

The dynamics of wave motion in the natural stratified medium (ocean, Earth atmosphere) is currently of great interest because they are important geophysics and oceanology. As a rule, an analysis of these phenomena is based on asymptotic methods, because the study of unperturbed hydrodynamic equations leads to asymptotic expansions (Ansatzs, which is a German term for a type of solutions). These expansions permit solving the problems of perturbed equations, which can be used to describe the effects of nonlinearity, inhomogeneity, and non-stationary behavior of the real ocean. To obtain a detailed description of a wide range of physical phenomena related to wave dynamics of the stratified horizontally inhomogeneous unsteady ocean, it is necessary to start from sufficiently developed mathematical models, which are usually quite complicated, nonlinear, and multi-parametric. They can be investigated completely only using efficient numerical methods. However, there are several cases, in which a preliminary qualitative concept of the phenomena under study can be obtained on the basis of simper asymptotic models and analytic methods for studying these models. These models then enter a set of «blocks» used to construct the complete pattern of wave dynamics, which permits discovering the correlation between different wave phenomena and their relationship. Sometimes, despite the seeming simplicity of the model assumptions, a successive choice of the solution form allows one to obtain physically interesting results [1, 2].

The propagation of internal gravity waves (IGW) in the ocean is strongly affected by the horizontal inhomogeneity and unsteady behavior of the basic hydro-physical parameters. In this contribution, we generalize a method of geometrical optics, i.e., the space-time ray method, which permits solving the

problem of mathematical modeling of IGW dynamics in the horizontally inhomogeneous and vertically stratified ocean. The ray representations agree well with the intuitive and empirical concepts of IGW propagation in the real ocean. This method is sufficiently universal, and in many cases, this is the only possible method for approximate calculations of wave fields in the ocean. The most typical horizontal inhomogeneities of the real ocean are the variations in the ocean floor shape, horizontal inhomogeneities of the density field, and unsteady ocean currents. An exact analytic solution can be obtained, for example, using the method of separation of variables only if the density distribution and the ocean floor shape can be described by sufficiently simple model functions. If the ocean floor shape and the ocean stratification are arbitrary, then one can construct only the asymptotic representations of the solution or solve the problem numerically. But the numerical solution does not permit obtaining and analyzing the qualitative characteristics of the wave field at large distances, which is necessary, for example, when solving the IGW detection problem by remote methods including, for example, the aerospace radiolocation.

The mathematical modeling of IGW wave dynamics in the horizontally inhomogeneous and vertically stratified medium (ocean, Earth atmosphere) is possible on the basis of a modified version of the space-time ray method (a method of geometrical optics, WKBJ method). The specific form of asymptotic representations can be determined by solving the problems, which describe the IGW dynamics in the vertically stratified, horizontally homogeneous, and steady-state ocean. As a rule, when studying the evolution of IGW packets in the ocean with slowly varying and unsteady parameters, it is assumed that this wave packet is locally harmonic. In contrast to the majority of works, in which this problem has been studied, the proposed modified method of geometrical optics allows one to describe the structure of wave packets near singular surfaces such as caustics and wave fronts. The term «geometrical optics» has different meanings in the scientific literature. The geometrical optics understood in the narrow (or ray) sense deals only with the methods for constructing images by using the rays, while the geometrical optics understood in the wide (or wave) sense is a method for obtaining approximate descriptions of wave fields. In the wave interpretation, which is used in this paper, the rays, as a rule, form only the geometric skeleton, on which the wave filed is «sewn on». According to the two previous interpretations of the geometrical optics, two periods in its development existed. The first ray period was ideologically completed by Hamilton's fundamental works, which significantly influenced the development of the classical mechanics. The construction of rays underlies the instrumental optics, which is mainly oriented to design various optical devices. The contemporary wave period originates from the Debye's works, which decisively influenced the formation of ray concepts in the wave theory. The asymptotic representation of solutions of the problem of wave packet propagation in the ocean with horizontally inhomogeneous density and numerical computations for typical oceanic parameters testify that the horizontal inhomogeneity significantly affects the real IGW dynamics in the ocean.

All results of wave dynamics modeling presented in this paper can be used for arbitrary density distributions and other parameters of the stratified ocean. It is necessary to consider them in the context of consistency with the available data of IGW full-scale measurements in the ocean. Such methods for analyzing the wave fields are important not only because they are illustrative, universal, and efficient in various problems, but also because they can serve as a certain semi-empirical basis for other approximate methods in the theory of wave packet propagation in the ocean. The waves in media with slowly varying parameters have been studied in the vast literature, while the amount of works dealing with the problem of studying IGW in the media with variable parameters is rather pure (mainly because of significant mathematical difficulties encountered in these problems). In this paper, we present the basics of the space-time ray method (a method of geometrical optics) with regard to IGW special characteristics, which permits studying the wave dynamics in the horizontally inhomogeneous and vertically stratified ocean [1,2].

2. Problem formulation

Our analysis starts from a linear system of hydrodynamic equations

$$\rho_0 \frac{\partial u_1}{\partial t} = -\frac{\partial p}{\partial x}, \qquad \rho_0 \frac{\partial u_2}{\partial t} = -\frac{\partial p}{\partial y}, \qquad \rho_0 \frac{\partial w}{\partial t} = -\frac{\partial p}{\partial z} + g\rho, \tag{1}$$

$$\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial w}{\partial z} = 0, \qquad \frac{\partial \rho}{\partial t} + u_1 \frac{\partial \rho_0}{\partial x} + u_2 \frac{\partial \rho_0}{\partial y} + w \frac{\partial \rho_0}{\partial z} = 0.$$

Here (u_1, u_2, w) are components of the IGW velocity vector; p and ρ are perturbations of the pressure and density; g is the acceleration of gravity (the axis z is directed downwards). Using the Boussinesq approximation, which means that the unperturbed density $\rho_0(z,x,y)$ in the first three equations in system (1) is assumed to be constant, we reduce system (1.1) to the form

$$\frac{\partial^4 w}{\partial z^2 \partial t^2} + \Delta \frac{\partial^2 w}{\partial t^2} + \frac{g}{\rho_0} \Delta \left(u_1 \frac{\partial \rho_0}{\partial x} + u_2 \frac{\partial \rho_0}{\partial y} + w \frac{\partial \rho_0}{\partial z} \right) = 0, \tag{2}$$

$$\frac{\partial}{\partial t}\left(\Delta u_1 + \frac{\partial^2 w}{\partial z \partial x}\right) = 0, \qquad , \quad \frac{\partial}{\partial t}\left(\Delta u_2 + \frac{\partial^2 w}{\partial z \partial y}\right) = 0 \qquad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

As the boundary conditions, we use the «rigid lid» condition at the surface: $W = 0$, *(z=0,-H)*, where H is the ocean depth. We assume that, in the media with horizontally inhomogeneous density field, the steady-state flows due to this field can be neglected. Indeed, it follows from the hydrodynamic equations that if the unperturbed density is a function of horizontal coordinates, then the existence of the steady-state density distribution $\rho_0(z,x,y)$ implies the existence of steady-state flows. These flows are rather slow, and they can be neglected in the first approximation. Therefore, it is usually assumed that $\rho_0(z,x,y)$ is the background density field formed under the action of mass forces and non-adiabatic sources, and this field is given *a priori*, for example, by experimental data.

Now we consider harmonic waves (u_1, u_2, w) = $exp(i\omega t)(U_1, U_2, W)$. System (2) cannot be solved by the method of separation of variables, and therefore it is necessary to use asymptotic methods. The scales of horizontal variations in the ocean parameters can be greater than the scales of vertical variability. Further we introduce the dimensionless variables: $x^* = x/L$, $y^* = y/L$, $z^* = z/h$, where L is the characteristic scale of horizontal variations in the density ρ_0 and h is the characteristic scale of vertical variations in ρ_0 (for example, the width of the thermocline). In the dimensionless coordinates, system (2) becomes (hereinafter, the asterisk in the indices is omitted)

$$-\omega^2 \left(\frac{\partial^2 W}{\partial z^2} + \varepsilon^2 \Delta W \right) + \varepsilon^2 \frac{g_1}{\rho_0} \left(\varepsilon U_1 \frac{\partial \rho_0}{\partial x} + \varepsilon U_2 \frac{\partial \rho_0}{\partial y} + W \frac{\partial \rho_0}{\partial z} \right) = 0, \tag{3}$$

$$\varepsilon \Delta U_1 + \frac{\partial^2 W}{\partial z \partial x} = 0, \qquad \varepsilon \Delta U_2 + \frac{\partial^2 W}{\partial z \partial y} = 0, \qquad \varepsilon = \frac{h}{L} << 1, \qquad g_1 = \frac{g}{h}.$$

3. Asymptotic forms of solution

We seek the asymptotic solution of (1.3) in the form typical of the method of geometrical optics

$$\mathbf{V}(z,x,y) = \sum_{m=0}^{\infty} (i\varepsilon)^m \mathbf{V}_m(z,x,y) \exp(S(x,y)/i\varepsilon), \ \mathbf{V}(z,x,y) = (U_1(z,x,y), U_2(z,x,y), W(z,x,y)),$$

where function $S(x,y)$ and vector function $\mathbf{V}_m, m$ = 0, 1…, are sought. As a rule, below, we determine only the leading term of this asymptotic expansion for the vertical velocity component $W_0(z,x,y)$. We obtain the following from the two last equations in (1.3)

$$U_{10} = -\frac{i\partial S/\partial x}{|\nabla S|^2}\frac{\partial W_0}{\partial z}, \qquad U_{20} = -\frac{i\partial S/\partial y}{|\nabla S|^2}\frac{\partial W_0}{\partial z}, \qquad |\nabla S| = \left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2.$$

Equating the terms of order O(1), we obtain the equation for function $W_0(z,x,y)$. This equation is written as

$$\frac{\partial^2 W_0(z,x,y)}{\partial z^2} + |\nabla S|^2\left(\frac{N^2(z,x,y)}{\omega^2} - 1\right)W_0(z,x,y) = 0, \quad W_0(0,x,y) = W_0(-H,x,y) = 0, \tag{4}$$

where $N^2(z,x,y) = \frac{g_1}{\rho_0}\frac{\partial \rho_0}{\partial z}$ is the Brunt–Väisälä frequency depending on the vertical and horizontal coordinates. It is well known that the basic boundary-value vertical spectral problem for internal waves (4) has countable many eigenfunctions W_{0n} and eigenvalues $K_n(x,y,\omega) \equiv |\nabla S_n|$. Functions $W_{0n}(z,x,y)$ and $K_n(x,y,\omega)$ are assumed to be known; the index n is omitted because we assume that all calculations are carried out for a separate wave mode. We use the eikonal equation $\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 = K^2(x,y)$ to determine function $S(x,y)$. In the plane case, the initial conditions for eikonal S are posed on line $L: x_0(\alpha), y_0(\alpha)$, $S(x,y)|_L = S_0(\alpha)$. To solve the eikonal equation, we construct the rays, i.e., the characteristics of this equation, which have the following form

$$\frac{dx}{d\sigma} = \frac{p}{K(x,y)}, \quad \frac{dp}{d\sigma} = \frac{\partial K(x,y)}{\partial x}, \quad \frac{dy}{d\sigma} = \frac{q}{K(x,y)}, \quad \frac{dq}{d\sigma} = \frac{\partial K(x,y)}{\partial y}, \tag{5}$$

where $p = \partial S/\partial x, \quad q = \partial S/\partial y, \quad d\sigma$ is the ray length element. The initial conditions p_0 and q_0 for solution (5) are determined by solving the following system

$$p_0\frac{\partial x_0}{\partial \alpha} + q_0\frac{\partial y_0}{\partial \alpha} = \frac{\partial S_0}{\partial \alpha}, \quad p_0^2 + q_0^2 = K^2\left(x_0(\alpha), y_0(\alpha)\right)$$

whose solution and the initial conditions $x_0(\alpha), y_0(\alpha), p_0(\alpha), q_0(\alpha)$ determine the ray $x = x(\sigma,\alpha), \; y = y(\sigma,\alpha)$. After the rays are constructed, eikonal S can be determined by integrating along the ray: $S = S_0(\alpha) + \int_0^\sigma K\left(x(\sigma,\alpha), y(\sigma,\alpha)\right)d\sigma$. Eigenfunction $W_0(z,x,y)$ is calculated up to multiplication by arbitrary function $A_0(x,y)$: $W_0(z,x,y) = A_0(x,y)\,f_0(z,x,y)$, where $f_0(z,x,y)$ is the solution of the basic vertical spectral problem with normalization

$$\int_0^H (N^2(z,x,y)-\omega^2) f_0^2(z,x,y)dz = 1.$$

Then, after rather cumbersome analytic calculations, we obtain the conservation law along the eikonal characteristics

$$\frac{d}{d\sigma}\left(ln\frac{A_0^2(x,y)I(x,y)}{K^2(x,y)} \right) = 0,$$

where $I(x,y)$ is the geometric divergence of the rays (characteristics). We note that the wave energy flux is proportional to $A_0^2 K^{-1} R$, where R is the width of an elementary ray tube; therefore, the quantity equal to the wave energy divided by the modulus of the wave vector is preserved in this case. The long-range IGW fields in the real ocean are, as a rule, non-harmonic wave packets. Indeed, at a far distance form perturbation sources, the complete wave field is a sum of separate wave modes whose asymptotics, depending on the stratification, depth, and other parameters of the ocean, can be expressed in terms of the Airy function or the Fresnel integrals. Therefore, to study the problem of wave packet evolution in a horizontally smoothly inhomogeneous and unsteady stratified medium, it is necessary to use another Ansatz.

We introduce slow variables $x^* = \varepsilon x,\ \ y^* = \varepsilon y,\ \ t^* = \varepsilon t$ (since z is not assumed to be a slow variable, we omit the asterisk in the index), where $\varepsilon = \lambda / L << 1$ is a small parameter characterizing the smoothness of the medium variations along the horizontal (λ is the characteristic wave length, and L is the scale of horizontal inhomogeneity). Then system (2) for determining the velocity components (U_1, U_2, W) in these slow variables becomes

$$\frac{\partial^4 W}{\partial z^2 \partial t^2} + \varepsilon^2 \frac{\partial^2 W}{\partial t^2} + \frac{g}{\rho_0}\Delta(\varepsilon U_1 \frac{\partial \rho_0}{\partial x} + \varepsilon U_2 \frac{\partial \rho_0}{\partial y} + W\frac{\partial \rho_0}{\partial z}) = 0, \qquad (6)$$

$$\varepsilon \Delta U_1 + \frac{\partial^2 W}{\partial z \partial x} = 0 \ \ , \ \varepsilon \Delta U_2 + \frac{\partial^2 W}{\partial z \partial y} = 0.$$

Further we consider the superposition of harmonic waves (in slow variables x, y, t) $W = \int \omega \sum_{m=0}^{\infty} (i\varepsilon)^m W_m(\omega, z, x, y)\, exp\left(\frac{i}{\varepsilon}[\omega t - S_m(\omega, x, y)] \right) d\omega$, where functions $S_m(\omega, x, y)$ are assumed to be odd with respect to ω and $\min_{\omega} \frac{\partial S}{\partial \omega}$ is attained at $\omega = 0$ (for all x and y). We substitute this representation into (6) and see that function $W_m(\omega, z, x, y)$ for $\omega = 0$ has a pole of order *m*. Therefore, the model integrals, or phase functions $R_m(\sigma)$, for some terms of the asymptotic series are expressions $R_m(\sigma) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (i/\omega)^{m-1} exp(i(\omega^3/3 - \sigma\omega)) d\omega$, where the contour of integration bypasses point $\omega = 0$ from above, which ensures the exponential decay of functions $R_m(\sigma)$ for $\sigma >> 1$. Functions $R_m(\sigma)$ have the following property $\frac{d R_m(\sigma)}{d\sigma} = R_{m-1}(\sigma)$, where $R_0(\sigma) = Ai'(\sigma)$,

$R_1(\sigma) = Ai(\sigma)$, $R_2(\sigma) = \int_{-\infty}^{\sigma} Ai(u)du$, etc. Obviously, starting from the corresponding properties of the Airy integrals, we can conclude that functions $R_m(\sigma)$ are related as $R_{-1}(\sigma) + \sigma R_1(\sigma) = 0$, $R_{-3}(\sigma) + 2R_0(\sigma) - \sigma^2 R_1(\sigma) = 0$. For the model integrals $R_m(\sigma)$ describing the long-range IGW fields in the deep ocean, one can use the following expressions $R_0(\sigma) = Re\int_0^{\infty} exp\left(-it\sigma - it^2/2\right)dt \equiv Re\,\Phi(\sigma)$; in this case, functions $R_m(\sigma)$ satisfy the recurrence relations $R_{-3}(\sigma) - 2iR_{-1}(\sigma) - i\sigma R_{-2}(\sigma) = 0$ and $R_{-1}(\sigma) + i\sigma R_0(\sigma) = 0$. It follows from the above and the structure of the first term of the uniform asymptotics (Airy or Fresnel wave) in a stratified and horizontally homogeneous medium that the solution of system (6) can be sought in the following form (index n is omitted for a separate wave mode)

$$W = \varepsilon^0 W_0(z,x,y,t)R_0(\sigma) + \varepsilon^a W_1(z,x,y,t)R_1(\sigma) + \varepsilon^{2a} W_2(z,x,y,t)R_2(\sigma) + \dots,$$
$$\mathbf{U} = \varepsilon^{1-a}\mathbf{U}_0(z,x,y,t)R_1(\sigma) + \varepsilon\mathbf{U}_1(z,x,y,t)R_2(\sigma) + \varepsilon^{1+a}\mathbf{U}_2(z,x,y,t)R_3(\sigma) + \dots,$$

where $\mathbf{U}$ is the vector of IGW horizontal velocity and the phase function argument $\sigma = \left(S(x,y,t)/a\right)^a \varepsilon^{-a}$ is assumed to be of the order of unity. This expansion agrees well with the general approach of the method of geometrical optics and the space-time ray method and is its generalization used to study the dynamics of IGW fields in the horizontally inhomogeneous stratified ocean. We also note that this structure of the solution implies that, in a horizontally inhomogeneous medium, the solution depends on both the «fast» (vertical coordinate) and «slow» (horizontal coordinates) variables. As a rule, the solution is sought in «slow» variables, and the structure elements depending on «fast» variables are obtained as integrals of some functions slowly varying along the space-time rays. This choice of the solution permits describing the uniform asymptotics of IGW fields propagating in the stratified ocean with slowly varying parameters, which is true both near and far from the wave fronts of a separate wave mode. If it is necessary to describe the behavior of the field only near the wave front, then one can use one of the methods of geometrical optics, i.e., the «traveling wave» method, and the weakly dispersion approximation in the form of the corresponding local asymptotics to seek the representation of the phase function argument σ in the form $\sigma = \alpha(t,x,y)(S(t,x,y) - \varepsilon t)\varepsilon^{-a}$; here function $S(t,x,y)$ describes the wave front position. It is found by solving the eikonal equation $\nabla^2 S = c^{-2}(x,y,t)$, where $c(t,x,y)$ is the maximal IGW group velocity of the corresponding wave mode, i.e., the first term in the expansion of the dispersion curve at zero. Function $\alpha(t,x,y)$ (the second term of the dispersion curve expansion) describes the space-time evolution of the pulse width of non-harmonic Airy or Fresnel waves and can then be found from some conservation laws along the eikonal equation characteristics whose specific form is determined by the physical conditions of the problems under study.

4. Comparison analytical and measurements results

Further we compare the analytic results with the results of the analysis of measurements of IGW variability in a real stratified medium with horizontally varying characteristics, namely, in the Northwest Pacific, according to the data recorded by moorings in the «Megapolygon» experiment in the Northwest Pacific. The measurements of the currents and the temperature recorded by the «Megapolygon» moorings allowed us to study the variability of tidal internal waves over the area of 460×520 km. The length of the tidal internal wave was calculated by integration of the basic IGW spectral equation with the real depth distribution of the Vaisala-Brunt frequency and with zero boundary conditions at the ocean surface and the ocean floor taking into account the Earth's rotation.

The wave length of the first mode in the «Megapolygon» area is equal approximately to 130 km, the wave length near the Emperor Ridge is greater (167 km), and it is equal to 156 km at a distance of 2000 km to the east. The wave propagation direction is also very stable and varies from 240 to 300 degrees, which corresponds to the actual wave propagation to the west and northwest from the Emperor Ridge. Some diffraction of tidal internal waves was observed in the «Megapolygon» study site, i.e., the direction of wave propagation varied from the northwest in the southeast of the site to the west in its northwest part. Let us consider the amplitude variations of the internal tide in the course of its propagation to the west and to the east from the Emperor Ridge. The IGW amplitudes were calculated from the deviations of the temperature values measured on moorings; then, the values were divided by the average vertical gradient of temperature. The calculations show that the IGW amplitude decreases approximately by 10% at the distance equal to the length of the tidal internal wave (130-150 km). We can also estimate the influence of different factors, including the horizontal inhomogeneity of density, on the IGW decay. In the framework of the theory discussed above, we consider the evolution of IGW frequency ω corresponding to the semidiurnal period $T = 12$ hours, which also admits slow variations in the stratification along the wave propagation path. The real geometry of the experiment allows us to assume that the problem under study is two-dimensional, which means that the stratification depends only on two variables: depth z and distance x along the wave propagation path [2].

Now we consider the case of constant depth H and stratification N linearly depending only on variable x: $N(x) = N_1 + (N_2 - N_1)x/L$, where L is the distance between the two observation points, $x = x_1 = 0$ is the initial point, $x = x_1 = L$ is the end point, and $N_{1,2} = N(x_{1,2})$. We consider only the first mode $\eta_1(z,x)$ of the amplitude of the vertical displacement of particles and omit its index. We seek the amplitude $\eta(z,x)$ in the form $\eta(z,x) = A(x)\, f(z,x)$, where $f(z,x)$ is the normalized eigenfunction of the standard boundary-value problem for the equation of internal waves with the normalization

$$\int_0^H (N^2(x) - \omega^2)\, f^2(z,x)\, dz = 1,$$

which has the form $f(z,x) = \sqrt{2}(H(N^2(x) - \omega^2))^{-1/2} \sin(\frac{\pi z}{H})$. Amplitude $A(x)$ depending only on x is determined from the conservation law: $\frac{A^2(x_1)}{k^2(x_1)} da(x_1) = \frac{A^2(x_2)}{k^2(x_2)} da(x_2)$, where $k(x)$ is the absolute value of horizontal wave vector, and $da(x)$ is the width of an elementary wave tube. Since the problem is two-dimensional, the width of the ray tube does not vary along the ray and the conservation law is simpler: $A(x)/k(x) = const$. Since we consider small values of ω, the velocity of wave propagation is close to the maximum group velocity $c(x) = N(x)H/\pi$; hence, the wave number is equal to $k(x) = \omega\pi / N(x)H$ and the corresponding wave length is equal to $\lambda(x) = 2N(x)H/\omega$. Then, under the assumption that the observation points are at the same depth, it follows from the conservation law $(A_{1,2} = A(x_{1,2}))$ that $A_1 N_1 = A_2 N_2$ or $A_2 = A_1 \lambda_1 / \lambda_2$. Then the total amplitude attains the following values $W_{1,2} = A_{1,2}\sqrt{2}(H(N_{1,2}^2 - \omega^2))^{-1/2}$, which implies $W_2 = W_1 N_1 ((N_1^2 - \omega^2)/(N_2^2 - \omega^2))^{1/2} / N_2$ or $W_2 = W_1 \lambda_1^2 \lambda_2^{-2}$, because $\omega << N$, i.e., the amplitude of the internal gravity wave is inversely proportional to the squared wave length. The wave travel time τ along the horizontal ray is determined from the equation of characteristics $\frac{dx}{dt} = c(x)$, where

$c(x)=(N_1+ax)H/\pi$ and $a=(N_2-N_1)/L$. Integrating this equation, we obtain the wave travel time $\tau=\pi ln(N_2/N_1)/aH=TL\,ln(\lambda_2/\lambda_1)/(\lambda_2-\lambda_1)$. The available data of full-scale tests give the following values of the basic parameters of the problem: $\lambda_1=167$ km, $\lambda_2=156$ km, $L=2000$ km. The wave attenuation coefficient without the wave length variations taken into account, which describes the amplitude decrease versus wave length, which is denoted by β, gives the value of $\beta:\beta=0.2^{167/2000}=0.874$ with regard to relation $W_2/W_1=0.2\equiv\beta^{t/T}=\beta^{L/\lambda}$ derived from the observation results. The attenuation with regard to the wave length variations along the ray, $W_2/W_1=\beta^{\tau/T}$, with the theoretically calculated time of the wave travel time τ gives the following value β = 0.878. Thus, the obtained estimates allow us to conclude that the influence of the density field inhomogeneities, which is taken into account in the above-described method for asymptotic representation of the wave fields, is one of the factors determining the scales of the space attenuation of IGW fields observed in field measurements.

5. Conclusions

Thus, in this paper, a general method for calculating IGW fields in the stratified medium outlined, namely:

– for an arbitrary distribution of the Brunt-Väisälä frequency, the basic vertical spectral IGW problem is solved and the corresponding normalized eigenfunctions and eigenvalues are determined;

– the characteristic systems with appropriate initial conditions are solved numerically;

– after the characteristics (rays) are calculated, the eikonal (phase value) of the phase functions is determined by numerical integration along these rays;

– the geometric divergence of the ray tubes is determined, for example, by numerical differentiation of closely located characteristics;

–the IGW amplitude is calculated from the equations of the corresponding conservation laws along the rays (characteristics), where the right parts of the relations are determined by using the locality principle, i.e., it is assumed that the ocean parameters remain horizontally unchanged over specific spatial intervals.

The universal character of the proposed asymptotic methods of modeling IGW fields in the stratified medium (ocean, Earth atmosphere) allows us to efficiently calculate the wave fields and, in addition, analyze qualitatively the obtained solutions. This opens wide opportunities for investigating the wave fields in general, which is also important for formulating correct statements of mathematical models of wave dynamics and for obtaining express evaluations in the field measurements of internal gravity waves [1, 2].

References

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