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Internal gravity waves from a moving source: modeling and asymptotics

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Internal gravity waves from a moving source: modeling and asymptotics

V V Bulatov[1] **and Yu V Vladimirov**[1]

[1] Ishlinsky Institute for Problems in Mechanics RAS, 119526 Moscow, Russia

E-mails: internalwave@mail.ru, vladimyura@yandex.ru

Abstract. In this paper we investigated the far internal gravity waves fields excited by a source of disturbances, moving in an infinite vertically stratified medium. The propagation of waves in an inviscid incompressible medium with an exponential distribution of unperturbed density is considered. In the linear approximation and the Boussinesq approximation, uniform asymptotics of the excited internal gravity waves fields were constructed far from the moving source of perturbations. Wave fields in the vicinity of the traverse plane and the horizon of motion are investigated. The obtained asymptotic solutions make it possible to effectively calculate the main amplitude-phase characteristics of the excited far internal gravity waves fields of under certain generation regimes. Analytical solutions allow to qualitatively analyze the solutions obtained. This is important for the correct formulation of more complex mathematical models of the wave dynamics of real natural stratified medium.

1. Introduction

An important mechanism for exiting fields of internal gravity waves (IGW) in natural (ocean, the Earth atmosphere) and artificial stratified media is their generation by sources of perturbation of various physical nature, i.e., of natural (moving typhoon, wind waves, flow past the ocean bottom relief imperfections, variations in the density and flow fields, leeward mountains) and anthropogenic (marine technological structures, collapse of the turbulent mixing region, underwater explosions) character. To obtain a solution of the system of hydrodynamic equations describing the wave perturbations of stratified media is generally a rather complicated mathematical problem from the standpoint of both the existence and uniqueness theorems for solutions in appropriate function classes and the computations. The main results obtained by solving the IGW generation problems are usually expressed in integral form, and the integral solutions obtained in this case require the development of numerical and asymptotic methods for their investigation, which permit qualitative analysis and express estimates of the obtained solutions [1-6].

To obtain a detailed description of a wide range of physical phenomena related to the dynamics of wave perturbations of inhomogeneous and nonstationary natural stratified media, it is necessary to start from sufficiently developed mathematical models. The fact that the structures of natural stratified media are three-dimensional also plays a significant role, and there is currently no possibility of performing full-scale computational experiments in simulation of three-dimensional ocean flows at large times with a sufficient accuracy. But in several cases, one can obtain the initial qualitative description of the wave phenomena in question by considering simpler analytic models. In this connection, it is necessary to mention the classical hydrodynamic problems of constructing asymptotic solutions that describe the evolution of wave perturbations generated by sources of various origination in a heavy liquid. The model solutions permit obtaining further descriptions of wave fields with regard to the variable and unsteady behavior of the real stratified media (ocean, the Earth atmosphere). Several results of analysis of model linear problems describing different regimes of generation and propagation of wave perturbations also underlie the currently actively developed nonlinear theory of generation of waves of extremely large amplitude, the rogue waves [7-11].

The IGW dynamics simulation is currently especially urgent because of the increasing number of off-shore structures located on shelf oil and gas fields. Several cases of off-shore structure damage by internal waves of large amplitude should be noted, for example, in the Andaman Sea when one of the platform legs was bent by the shear flow in the internal wave in October 1997. The measurements show that the loads due to the IGW acting on the underwater parts of the off-structures in the vertical direction can be 30 times greater than the loads due to the wind waves. The action of IGW waves results in the powerful transport of deposition and the bed movement, especially in deep water regions, where the influence of wind waves, including the storm waves, is negligibly small. The IGW also facilitate the sediment diffusion and the deposition transport in the marine environment, and hence the processes of particle transport by the wave-induced flows are actively studied in different applied fields related to the hydrobiology (plankton and benthos migration), ecology (propagation of admixtures and impurities), and engineering oceanology [8-10].

The IGW existing in the ocean are in principle two-dimensional due to the stratification of their waters and three-dimensional in many cases; therefore, the computational analysis of two- and three-dimensional non-stationary wave motions is a very complicated problem. The MIT numerical code for solving the complete hydrodynamic equations with regard to the real bottom relief, the Earth rotation, and turbulent processes was developed at Massachusetts Institute of Technology (USA) in cooperation with specialists in the ocean numerical simulation from the world community and has been used widely nowadays. This model requires a large amount of computer resources, which can be justified only when it is necessary to solve several practical problems of oceanology. Nevertheless, even such complete models do not still take into account, for example, the stable background horizontally inhomogeneous stratification that really exists in the ocean. To take this hydrophysical effect into account, it is necessary to introduce external forces that maintain this stratification inhomogeneity, but it is rather difficult to parameterize such forces numerically. The other currently existing methods for numerical modelling, including the methods based on the use of supercomputers (IGW Research algorithm, Riemann Solver algorithms for solving hyperbolic equations of shallow water, higher-order pseudo-spectral algorithm for solving HOSM hydrodynamic equations) do not always permit efficiently calculating specific physical problems of wave dynamics of the ocean and atmosphere with their actual variability taken into account, because they are oriented to solve rather general problems that require large computational power and do not always take into account the physical nature of the problems under study, which significantly restricts their practical applicability, especially in calculations of wave fields in real natural environments. Moreover, the use of powerful numerical algorithms requires verification and comparison with the solutions of model problems [9, 11-16].

Therefore, in the contemporary research, simplified asymptotic and analytic models are widely used to analyse the wave phenomena in real stratified media (ocean, the Earth atmosphere). In the linear approximation, the existing approaches to the description of the wave pattern of generated IGW fields are based on the representation of wave fields by Fourier integrals. The goal in the present paper is to study the far IGW fields generated by a perturbation source moving in a vertically infinite stratified medium.

2. Problem formulation, integral forms of solutions

We consider the IGW propagation in an inviscid incompressible medium with exponential distribution of unperturbed density $\rho_0(z)=exp(-\lambda z)$, i.e., we assume that the Brunt-Vaisaala frequency $N^2(z)=-\dfrac{g}{\rho_0(z)}\dfrac{d\rho_0(z)}{dz}$ is constant, $N(z)=\sqrt{\lambda g}=N=const$. In the linear approximation with regard to the Boussinesq approximation, the IGW velocity components (u,v,w) satisfy the system of equations [3,6,10,11]

$$Lw=-\frac{\partial^2}{\partial t\partial z}\left[\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}-\frac{\partial M}{\partial t}\right]+\Delta\frac{\partial F_z}{\partial t} \quad (1)$$

$$\frac{\partial Lu}{\partial t}=\frac{\partial^2}{\partial t^2}\left[\frac{\partial^2 F_x}{\partial y^2}+\frac{\partial^2 F_x}{\partial z^2}\right]+N^2\frac{\partial^2 F_x}{\partial y^2}-\left(\frac{\partial^2}{\partial t^2}+N^2\right)\frac{\partial^2 F_y}{\partial x\partial y}-\frac{\partial^4 F_z}{\partial t^2\partial x\partial z}+\left(\frac{\partial^2}{\partial t^2}+N^2\right)\frac{\partial M}{\partial x\partial t}$$

$$\frac{\partial Lv}{\partial t}=\frac{\partial^2}{\partial t^2}\left[\frac{\partial^2 F_y}{\partial x^2}+\frac{\partial^2 F_y}{\partial z^2}\right]+N^2\frac{\partial^2 F_y}{\partial x^2}-\left(\frac{\partial^2}{\partial t^2}+N^2\right)\frac{\partial^2 F_x}{\partial x\partial y}-\frac{\partial^4 F_z}{\partial t^2\partial y\partial z}+\left(\frac{\partial^2}{\partial t^2}+N^2\right)\frac{\partial^2 M}{\partial y\partial t}$$

$$L=\frac{\partial^2}{\partial t^2}(\Delta+\frac{\partial^2}{\partial z^2})+N^2\Delta,\quad \Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$$

where F_x, F_y, F_z are the mass forces and M is the distribution density of mass sources. In what follows, we consider the IGW fields excited by a nonlocal source of perturbations that uniformly rectilinearly moves with a constant velocity V along the axis Ox. Such a field is usually approximated as the field generated by a system of moving sources; in the presence of the buoyancy force, it is necessary to consider F_z and put the mass force components F_x, F_y equal to zero. As a result, system (1) can be represented as

$$Lw=\frac{\partial^3 M}{\partial z\partial t^2}+\Delta\frac{\partial F_z}{\partial t},\ Lu=-\frac{\partial^3 F_z}{\partial t\partial x\partial z}+\left(\frac{\partial^2}{\partial t^2}+N^2\right)\frac{\partial M}{\partial x},\ Lv=-\frac{\partial^3 F_z}{\partial t\partial y\partial z}+\left(\frac{\partial^2}{\partial t^2}+N^2\right)\frac{\partial M}{\partial y} \quad (2)$$

If the intermediate processes related to the beginning of the perturbation source motion are neglected, then the right-hand sides of (2) are functions of $\xi=x+Vt$, y,z. Since the problem is linear, the solutions of equations (2) are expressed in terms of the Green function G, i.e., in terms of the solution of the equation $LG(x+Vt,y,z)=\delta(x+Vt)\delta t)\delta(y))$. The solution η of the equation $L\eta=\Pi$ with zero right-hand side Π is expressed in terms of the Green function G by the formula $\eta(\xi,y,z)=\iiint G(\xi-\gamma, y-\alpha, z-\beta)\,\Pi(\gamma,\alpha,\beta)\,d\gamma\, d\alpha\, d\beta$. The uniqueness of the solution G is ensured by the radiation condition, i.e., one considers the solution of the equation with the right-hand side exponentially increasing with time: $LG_\varepsilon(t,x,y,z)=exp(\varepsilon t)\delta)\delta+Vt,y,z)$, and it is assumed that G_ε exhibits the same increase in t: $G_\varepsilon=exp(\varepsilon t)G_1(x+Vt,y,z)$. Then G is constructed uniquely and the solution is determined as the limit of G_ε as $\varepsilon\to 0$. Applying the operator L to the function G_ε, we obtain the equation for G: $\left(\varepsilon+V\frac{\partial}{\partial\xi}\right)^2\left[\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right]G+N^2\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial y^2}\right)G=\delta(\xi,y,z)$.

Solving this equation by the Fourier method, we obtain

$$G=\frac{1}{(2\pi)^2}\iiint\frac{exp(i(\alpha i+\beta y+\gamma z))d\alpha\, d\beta\, d\gamma}{(\alpha V-i\varepsilon)^2((\alpha^2+\beta^2+\gamma^2)-N^2(\alpha^2+\beta^2))}$$

Let us calculate the integral over γ under the assumption that $z>0$ (one can easily see that G is an even function of z). This integral is equal to the residual of the integrand pole in the upper half-plane. For $\alpha^2V^2<N^2$, this pole lies at the distance of the order of ε from the real axis:

$\gamma = (\sqrt{\alpha^2+\beta^2}\ \sqrt{N^2-\alpha^2V^2+2i\alpha\varepsilon V}\ )/(\alpha V - i\varepsilon)$. For $\alpha^2V^2 > N^2$, this pole lies near the imaginary axis: $\gamma = i\,(\sqrt{\alpha^2V^2-N^2-2i\alpha\varepsilon V}\ \sqrt{\alpha^2+\beta^2}\ )/(\alpha V - i\varepsilon)$. Calculating the corresponding integral and passing to the limit as $\varepsilon \to 0$, we obtain the sought function in the form

$$G = \frac{i}{8\pi^2V^2}\iint \frac{exp(\,i(\alpha\xi+\beta y+\sqrt{\chi^2-\alpha^2}\ \sqrt{\alpha^2+\beta^2}\,|z|/\alpha\,))\,d\alpha\, d\beta}{\alpha\sqrt{\alpha^2+\beta^2}\ \sqrt{\chi^2-\alpha^2}}\ ,\ \chi^2 = N^2/V^2$$

Here $\sqrt{\chi^2-\alpha^2}$ is the arithmetic value of the root for $|\alpha| < \chi$ and $i\sqrt{\alpha^2-\chi^2}$ for $|\alpha| > \chi$; when we integrate over α, the pole is bypassed in the lower half-plane. Further, we consider how the elevation ζ ($w = \frac{\partial\zeta}{\partial t}$) and the velocity components (u,v) can be expressed in terms of the function G in the case of moving δ -shaped source of mass and dipole. Assume that $F_z = 0$, $M = \delta(\xi,y,z)$, $\xi = x + Vt$. Then the velocity components (u,v,w) can be expressed as: $Lw = V^2\frac{\partial^3\delta(\xi,y,z)}{\partial\xi\partial\xi\partial z}$, $Lu = \left(V^2\frac{\partial^2}{\partial\xi^2}+N^2\right)^2\frac{\partial\delta(\xi,y,z)}{\partial\xi}$, $Lv = \left(V^2\frac{\partial^2}{\partial\xi^2}+N^2\right)^2\frac{\partial\delta(\xi,y,z)}{\partial y}$. For the elevation ζ we have: $L\zeta = V\frac{\partial^2\delta(\xi,y,z)}{\partial\xi\partial z}$. Therefore, in the case of moving point source of mass, the elevation $\zeta = \zeta_M$ and the velocity components $u = u_M$, $v = v_M$ are equal to

$$\zeta_M = V\frac{\partial^2 G}{\partial\xi\partial z} = \frac{-isign\,z}{8\pi^2V}\iint\frac{exp(i(\alpha\xi+\beta y+\gamma|z|))d\alpha d\beta}{\alpha}$$

$$u = u_M = V^2\left(\frac{\partial^2}{\partial\xi^2}+\chi^2\right)\frac{\partial G}{\partial\xi} = -\frac{1}{8\pi^2}\iint\sqrt{\frac{\chi^2-\alpha^2}{\alpha^2+\beta^2}}\ exp(i(\alpha\xi+\beta y+\gamma|z|))d\alpha d\beta$$

$$v = v_M = V^2\left(\frac{\partial^2}{\partial\xi^2}+\chi^2\right)\frac{\partial G}{\partial y} = -\frac{1}{8\pi^2}\iint\frac{\beta}{\alpha}\sqrt{\frac{\chi^2-\alpha^2}{\alpha^2+\beta^2}}\ exp(i(\alpha\xi+\beta y+\gamma|z|))d\alpha d\beta$$

where $\gamma = \sqrt{\chi^2-\alpha^2}\sqrt{\alpha^2+\beta^2}\,/\,\alpha$. For the dipole source of mass that is oriented along the source motion and has the unit moment, we obtain

$$\zeta = \zeta_D = \frac{\partial\zeta_M}{\partial\xi} = \frac{sign\,z}{8\pi^2V}\iint exp(i(\alpha\xi+\beta y+\gamma|z|))d\alpha d\beta$$

$$u = u_D = \frac{\partial u_M}{\partial\xi} = -\frac{i}{8\pi^2}\iint\frac{\alpha\sqrt{\chi^2-\alpha^2}}{\sqrt{\alpha^2+\beta^2}}\ exp(i(\alpha\xi+\beta y+\gamma|z|))d\alpha d\beta$$

$$v = v_D = \frac{\partial v_M}{\partial \xi} = -\frac{i}{8\pi^2}\iint \frac{\beta\sqrt{\chi^2-\alpha^2}}{\sqrt{\alpha^2+\beta^2}} exp(i(\alpha\xi+\beta y+\gamma|z|))d\alpha d\beta$$

Further, we put

$$\Gamma = \frac{\partial G}{\partial \xi} = \frac{-1}{8\pi^2 V^2}\iint \frac{exp(i(\alpha\xi+\beta y+\gamma|z|))d\alpha d\beta}{\sqrt{\alpha^2+\beta^2}\sqrt{\chi^2-\alpha^2}} \tag{3}$$

Then we have: $\zeta_D = V\frac{\partial^2 \Gamma}{\partial\xi\partial z}$, $u_D = V^2\left(\frac{\partial^2}{\partial\xi^2}+\chi^2\right)\frac{\partial\Gamma}{\partial\xi}$, $v_D = V^2\left(\frac{\partial^2}{\partial\xi^2}+\chi^2\right)\frac{\partial\Gamma}{\partial y}$

3. Asymptotic solutions

Further, we consider the asymptotics of the integral (3) for $r = \sqrt{\xi^2 + y^2 + z^2} >> 1$, i.e., the IGW field at a far distance from the moving source of perturbations. Since the phase function in (3) is an odd function of the variables α, β, we have

$$\Gamma = \frac{-1}{4\pi^2 V^2} Re\int_0^\infty d\alpha \int_{-\infty}^\infty d\beta \, \frac{exp(i(\alpha\xi+\beta y+\sqrt{\chi^2-\alpha^2}\sqrt{\alpha^2+\beta^2}|z|/\alpha))}{\sqrt{\alpha^2+\beta^2}\sqrt{\chi^2-\alpha^2}}$$

In this integral, we change the variables by the formulas $\alpha = \chi p$, $\beta = \chi pq$ and put $\xi = rcos\theta$, $y = rsin\theta cos\varphi$, $|z| = rsin\theta sin\varphi$ $(0<\theta,\ \varphi<\pi)$, i.e., we pass to the spherical coordinates r,θ,φ. Then we obtain

$$\Gamma = Re\Phi \tag{4}$$

$$\Phi = \frac{-1}{4\pi^2 V^2}\int_0^\infty \frac{dp}{\sqrt{1-p^2}}\int_{-\infty}^\infty \frac{exp(i\chi r(pcos\theta + pqsin\theta cos\varphi + \sqrt{1-p^2}\sqrt{1+q^2}sin\theta sin\varphi))dq}{\sqrt{1+q^2}}$$

The expression $\sqrt{1-p^2}$ for $p>1$ is understood as $i\sqrt{p^2-1}$, i.e., we bypass the cut of the function $\sqrt{1-p^2}$ as $Rep\to\infty$ in the lower half-plane. First, we determine the inhomogeneous asymptotics of Γ, i.e., the asymptotics that can be applied as $r\to\infty$ for fixed values of θ,φ such that $\theta \neq 0,\pi/2,\pi,\ \varphi\neq 0$. Calculating the inner integral in (4) by the stationary phase method, we obtain [6,18]

$$\Phi \approx -\frac{exp(i\pi/4)}{(2\pi)^{3/2}V^2\sqrt{\chi r}}\int_0^\infty \frac{exp(i\chi r(pcos\theta+\sqrt{sin^2\varphi-p^2}sin\theta))\,dp}{\left(sin^2\varphi-p^2\right)^{1/4}\sqrt{sin\theta}\sqrt{1-p^2}} \tag{5}$$

We write the inhomogeneous asymptotics of the integral (5). It consists of the following two terms: the contribution of the boundary of the domain of integration and the contribution of the stationary point of the phase function in (5). Since this point $p = p_0 = \sin\varphi \cos\theta$ is located in the domain of integration and makes a contribution to the asymptotics only for $\theta < \pi/2$, we obtain

$$\Phi \approx -\frac{exp(i\chi rsin\varphi)\ \Theta(cos\theta)}{2\pi V^2 \chi r\sqrt{1-sin^2\varphi\ cos^2\theta}} + \frac{exp(i\pi/4)exp(i\chi rsin\varphi sin\theta)}{(2\pi)^{3/2} V^2 (\chi r)^{3/2}\sqrt{sin\varphi\ sin\theta}\ cos\theta} \tag{6}$$

where $\Theta(cos\theta) = 1$ for $cos\theta > 0$ and $\Theta(cos\theta) = 0$ for $cos\theta < 0$. The asymptotics (6) cannot be used, first, as $cos\theta \to 0$ (i.e., for small $\xi = x + Vt$ near the traverse plane of motion of the perturbation source) and, second, as $sin \quad sin\theta \to 0$, i.e., for small $|z| = \chi rsin\varphi sin\theta$ near the horizontal plane of the source motion. Further, we consider the asymptotics that can be applied near the transverse plane. As $cos\theta \to 0$, the stationary point of the phase function $p_0 = sin\varphi cos\theta$ tends to the boundary of the domain of integration $p = 0$. There is a known algorithm for determining the asymptotics of the integral (4) that is uniform in $cos\theta$, and this asymptotics can be expressed in terms of the Fresnel integral. To obtain this asymptotics, it is necessary to replace the function $\Theta(cos\theta)$ in (6) by the expression

$$F^*\left(\sqrt{2\chi rsin\varphi} sin(\pi/4 - \Theta/2)\right) + \frac{exp(i\pi/4 - 2i\chi rsin\varphi\ sin^2(\pi/4 - \theta/2))}{2\sqrt{\pi}\ \sqrt{2\chi rsin\varphi}\ \ sin(\pi/4 - \theta/2)}$$

where $F^*(\tau)$ is the complex conjugate Fresnel integral [18,19]: $F^*(\tau) = \frac{exp(i\pi/4)}{\sqrt{\pi}} \int_{-\infty}^{\tau} exp(-is^2)ds$

Then we have

$$\Gamma = Re\Phi \approx Re\,(-\frac{exp(i\chi rsin\varphi)\ F^*\left(\sqrt{2\chi rsin\varphi}\ sin((\pi/4 - \theta/2))\right)}{2\pi V^2 \chi r\sqrt{1-sin^2\varphi cos^2\theta}} + \tag{7}$$

$$+\frac{exp(i\chi rsin\varphi sin\theta + i\pi/4)(\sqrt{1-sin^2\varphi cos^2\theta} - sin((\pi/4-\theta/2))\sqrt{sin\theta})}{V^2(2\pi)^{3/2}(\chi r)^{3/2}\sqrt{sin\varphi sin\theta}\sqrt{1-sin^2\varphi cos^2\theta}\ cos\theta})$$

The obtained expression (7) can be used for θ close to $\pi/2$ but cannot be used as $\sin\phi\sin\theta \to 0$, i.e., near the horizontal plane of the source motion. Further, we transform the integral (4) over q by putting $q = sht$: $F(a,b) = \int_{-\infty}^{\infty} \frac{exp(i(aq + b\sqrt{1+q^2}))dq}{\sqrt{1+q^2}} = \int_{-\infty}^{\infty} exp(i(a\ sh + b\ ch\ t))\ dt$.

By the shift in t in the obtained integral, we can easily show that $F(a,b) = F(a_1, b_1)$ for $a^2 - b^2 = a_1^2 - b_1^2$. Therefore, the integral over q in (4) can be written as

$$\int_{-\infty}^{\infty} \frac{exp(i\chi rsin\theta(pqcos\varphi + \sqrt{1-p^2}\ \sqrt{1+q^2}sin\varphi))dq}{\sqrt{1+q^2}} = \tag{8}$$

$$= \int_{-\infty}^{\infty} \frac{exp(i\chi rsin\theta(pq + \sqrt{1+q^2}sin\varphi))dq}{\sqrt{1+q^2}} = \int_{-\infty}^{\infty} \frac{exp(i\chi rsin\theta(qcos\varphi + \sqrt{1-p^2}\sqrt{1+q^2}))dq}{\sqrt{1+q^2}}$$

We use the first relation in (8) to write Φ as

$$\Phi = \frac{-1}{4\pi^2 V^2} \int_0^{\infty} \frac{dp}{\sqrt{1-p^2}} \int_{-\infty}^{\infty} \frac{exp(i\chi r(p(cos\theta + q\,sin\theta) + \sqrt{1+q^2}\,sin\theta\,sin\varphi))dq}{\sqrt{1+q^2}} \tag{9}$$

In the integral (9), just as in (4), the expression $\sqrt{1-p^2}$ for $p>1$ is understood as $i\sqrt{p^2-1}$, i.e., we bypass the cut of the function $\sqrt{1-p^2}$ as $Re p \to \infty$ in the lower half-plane. Further, we shift the contour of integration with respect to the variable q upwards from the real axis by the value i and for $Re q > 0$, rotate the contour by a certain angle $\phi (0 < \phi < \pi/2)$ anticlockwise. With respect to the variable p for $Re p > 0$, we rotate the contour of integration by a certain angle $\vartheta (0 < \vartheta < \pi/2)$ clockwise, so that $\phi < \vartheta$. Then for $cos\theta \le 0$ and such a choice of the contours of integration, the exponent in (9) has negative real part, the integral absolutely converges as $p \to \infty$, $|q| \to \infty$, and the order of integration in (9) can be changed

$$\Phi = \frac{-1}{4\pi^2 V^2} \int_{-\infty+i}^{\infty\, exp(i\phi)} \frac{exp(i\chi r\sqrt{1+q^2}\,sin\theta sin\varphi)dq}{\sqrt{1+q^2}} \int_0^{\infty\, exp(-i\vartheta)} \frac{exp(i\chi r(cos\theta + qsin\theta)p)dp}{\sqrt{1-p^2}}$$

Now the inner integral can asymptotically be calculated for $\chi r >> 1$ as

$$\int_0^{\infty} \frac{exp(i\chi r(cos\theta + qsin\,\theta)p)dp}{\sqrt{1-p^2}} = \frac{i}{\chi r(cos\theta + qsin\,\theta)} + O(\chi r)^{-3}$$

and for $\Gamma = Re\,\Phi$, we obtain

$$\Gamma = \frac{-1}{4\pi^2 V^2 \chi r} Im \int_{-\infty}^{\infty} \frac{exp(i\chi r\sqrt{1+q^2}\,\,sin\theta sin\varphi)\,dq}{\sqrt{1+q^2}\,\,(cos\theta + qsin\,\theta)} + O(\chi r)^{-3} \tag{10}$$

where the pole for $q = -ctg\,\theta$ is bypassed in the lower half-plane. This function can be written as $(rsin\,\theta\,sin\varphi = z)$

$$\Gamma \approx -\frac{1}{4\pi^2 V^2 \chi r} Im \frac{1}{2} \int_{-\infty}^{\infty} \frac{exp(i\chi z\sqrt{1+q^2}\,)}{\sqrt{1+q^2}} \left(\frac{1}{cos\,\theta + qsin\,\theta} + \frac{1}{cos\,\theta - qsin\,\theta} \right) dq =$$

$$= \frac{-cos\,\theta}{4\pi^2 V^2 \chi r} Im \int_{-\infty}^{\infty} \frac{exp(i\chi z\sqrt{1+q^2}\,)}{\sqrt{1+q^2}} \frac{dq}{cos^2\theta - q^2 sin^2\theta}$$

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where the pole $q=-ctg\,\theta$ is bypassed in the upper half-plane, and the pole $q=ctg\,\theta$, in the lower half-plane. Let us prove that

$$cos\,\theta\,Im\int_{-\infty}^{\infty}\frac{exp(i\tau\sqrt{1+q^2}\,)}{\sqrt{1+q^2}}\frac{dq}{cos^2\theta-q^2sin^2\theta}=-2\pi\left[\frac{J_0(\tau)}{2}+\sum_{n=1}^{\infty}\left(-ctg^2\frac{\theta}{2}\right)^nJ_{2n}(\tau)\right]\qquad(11)$$

where $J_{2n}(\tau)$ are Bessel functions [19-21]. Indeed, we denote the left-hand side of (11) by $F(\tau(\theta)$ and obtain

$$F(\tau,\theta)+sin^2\theta\,\frac{\partial^2F(\tau,\theta)}{\partial\tau^2}=cos\theta\,Im\int_{-\infty}^{\infty}\frac{exp(i\tau\sqrt{1+q^2}\,)dq}{\sqrt{1+q^2}}=cos\theta\,Im(i\pi H_0^{(1)}(\tau))=\pi cos\theta J_0(\tau)$$

Here $H_0^{(1)}(\tau)$ is the Hankel function [19-21]. Therefore, $F(\tau,\theta)$ can be determined as the solution of the equation: $F(\tau,\theta)+\sin^2\theta\,\dfrac{\partial^2F(\tau,\theta)}{\partial\tau^2}=\pi\cos\theta\,J_0(\tau),$ that tends to zero as $|\tau|\to\infty$ like $|\tau|^{-1/2}$, which follows from the above-formulated rule for bypassing the poles in the integral (10). Then formula (11) can be verified directly by using the recursive formulas for the Bessel functions [21]: $\dfrac{\partial^2J_0(\tau)}{\partial\tau^2}=\dfrac{1}{2}(J_2(\tau)-J_0(\tau)),\quad \dfrac{\partial^2J_{2n}(\tau)}{\partial\tau^2}=\left(\dfrac{J_{2n-2}(\tau)}{4}-\dfrac{J_{2n}(\tau)}{2}+\dfrac{J_{2n+2}(\tau)}{2}\right)$. Thus, for $\chi r>>1$ and $cos\,\theta\le 0$, we obtain the following asymptotic approximation for Γ in the form

$$\Gamma=\frac{1}{2\pi V^2\chi r}\left(\frac{J_0(\chi z)}{2}+\sum_{n=1}^{\infty}\left(-ctg^2\frac{\theta}{2}\right)^nJ_{2n}(\chi z)\right)+O(\chi r)^{-3}\qquad(12)$$

where $z=2sin\,\theta\,sin\varphi$. It is convenient to use this formula for small χz, because $J_{2n}(\chi z)$ exponentially tend to zero for $2n>\chi z$ and it suffices to take relatively few terms in the power series in $-ctg^2\dfrac{\theta}{2}$. Moreover, an additional factor of convergence is the multiplier $ctg^2\dfrac{\theta}{2}$ that vanishes for $\theta=\pi$. For $\chi z>>1$ and $cos\,\theta<0$, formula (12) is asymptotically equivalent to (7). Indeed, for $cos\,\theta$ different from zero, $cos\,\theta<const<0$ and $\chi z>>1$, the asymptotics of the integral (10) is determined by the stationary point $q=0$ and coincides with the second term in (6) (the first term is zero for $cos\,\theta<0$). If $cos\,\theta$ is close to zero, then it is necessary to use the asymptotics of the integral (10) that is uniform in $cos\,\theta$, i.e., with respect to the distance between the pole $q=-ctg\,\theta$ and the stationary point $q=0$. This asymptotics can be expressed in terms of the complex conjugate Fresnel integral and has the form

$$\Phi=-\frac{exp(i\chi rsin\varphi)}{2\pi V^2\chi r}T\left(\sqrt{2\chi rsin\varphi}\,sin\left(\pi/4-\theta/2\right)\right)+\frac{exp(i\pi/4+i\chi rsin\varphi sin\theta)}{(2\pi)^{3/2}V^2(\chi r)^{3/2}\,cos\theta\sqrt{sin\varphi sin\theta}}$$

$$T(\rho) = F^*(\rho) + \frac{exp(i\pi/4 - i\rho^2)}{2\sqrt{\pi}\ \rho},\ \rho = \sqrt{2\chi r sin\varphi}\ sin\left(\pi/4 - \theta/2\right)$$

i.e., differs from (7) by the term

$$\frac{exp(i\chi r sin\varphi)}{2\pi V^2 \chi r} \frac{1 - \sqrt{1 - sin^2\varphi cos^2\theta}}{\sqrt{1 - sin^2\varphi cos^2\theta}} T(\rho) \tag{13}$$

We now show that this expression is small for $\chi r >> 1$. For this, we consider the function $\rho^2 T(\rho)$. Since $T(\rho) = O(\rho^{-3})$ as $\rho \to -\infty$ and $\rho^2 T(\rho) \to 0$ as $\rho \to 0$, the function $\rho^2 T(\rho)$ is bounded for $\rho < 0$: $\left|\rho^2 T(\rho)\right| < M$, where M is a certain fixed number. Therefore, the expression (13) can be rewritten as

$$\frac{exp(i\chi r sin\varphi)}{2\pi V^2 \chi r} \frac{1 - \sqrt{1 - sin^2\varphi cos^2\theta}\ \rho^2 T(\rho)}{\sqrt{1 - sin^2\varphi cos^2\theta}\ 2\chi r\ sin\varphi sin^2\left(\pi/4 - \theta/2\right)}$$

Since the function $\rho^2 T(\rho)$ is bounded and, for $-1 < const < cos\theta < 0$, the expression

$$\frac{1 - \sqrt{1 - sin^2\varphi cos\theta}}{\sqrt{1 - sin^2\varphi cos^2\theta}\ sin\varphi\ sin^2\left(\pi/4 - \theta/2\right)} = \frac{2sin\varphi \quad (1 + sin\theta)}{\sqrt{1 - sin^2\varphi cos^2\theta}(1 + \sqrt{1 - sin^2\varphi cos^2\theta})}$$

is also bounded, we see that the function (13) is of the order of $(\chi r)^{-2}$.

Now we consider the values $\cos\theta > 0$. After the change $p, q \to -p, -q$, the integral (4) takes the form

$$\Phi(r,\theta,\varphi) = \frac{-1}{4\pi^2 V^2} \int_{-\infty}^{0} \frac{dp}{\sqrt{1 - p^2}} \int_{-\infty}^{\infty} \frac{exp(iA)dq}{\sqrt{1 + q^2}}$$

$$A = \chi r(-pcos\theta + pq\ sin\ \theta sin\varphi + \sqrt{1 - p^2}\ \sqrt{1 + q^2}\ sin\theta sin\varphi)$$

On the other hand, it is obvious that

$$\Phi(r,\pi - \theta,\varphi) = \frac{-1}{4\pi^2 V^2} \int_{0}^{\infty} \frac{dp}{\sqrt{1 - p^2}} \int_{-\infty}^{\infty} \frac{exp(iA)dq}{\sqrt{1 + q^2}}$$

$$\Phi(r,\theta,\varphi) + \Phi(r,\pi - \theta,\varphi) = \Lambda(r,\theta,\varphi) = \frac{-1}{4\pi^2 V^2} \int_{-\infty}^{\infty} \frac{dp}{\sqrt{1 - p^2}} \int_{-\infty}^{\infty} \frac{exp(iA)dq}{\sqrt{1 + q^2}}$$

$$\Gamma(r,\theta,\varphi) = -\Gamma(r,\pi - \theta,\varphi) - \frac{Re\Lambda(r,\theta,\varphi)}{4\pi^2 V^2}$$

Since the asymptotics of $\Gamma(r,\theta,\varphi)$ for $\theta > \pi/2$ is already known, it suffices to obtain the asymptotics of the integral Λ. Using the second relation in (8), we can write the expression for Λ in the form

$$\Lambda = \frac{-1}{4\pi^2 V^2} \int_{-\infty}^{\infty} \frac{exp(-i\chi rpcos\theta)dp}{\sqrt{1-p^2}} \int_{-\infty}^{\infty} \frac{exp(i\chi rsin\theta\left(qcos\varphi + \sqrt{1-p^2}\ \sqrt{1+q^2}\right))dq}{\sqrt{1+q^2}}$$

and changing the order of integration, we obtain

$$\Lambda = \frac{-1}{4\pi^2 V^2} \int_{-\infty}^{\infty} \frac{exp(i\chi rqsin\theta cos\varphi)dq}{\sqrt{1+q^2}} \int_{-\infty}^{\infty} \frac{exp(i\chi r(-pcos\theta + \sqrt{1-p^2}\ \sqrt{1+q^2}\,sin\,\theta))dp}{\sqrt{1-p^2}} \tag{14}$$

The asymptotics of the inner integral for $\chi r >> 1$ is determined by the stationary phase method and turns out to be uniform with respect to q,θ,φ. As a result, we obtain

$$\Lambda \approx -\frac{exp(i\pi/4)}{(2\pi)^{3/2}V^2\sqrt{\chi r}} \int_{-\infty}^{\infty} \frac{exp(i\chi r(qsin\,\theta cos\varphi + \sqrt{1+q^2 sin^2\theta}\,))}{(1+q^2 sin^2\theta)^{1/4}\sqrt{1+q^2}}\, dq = \tag{15}$$

$$= -\frac{exp(-i\pi/4)}{(2\pi)^{3/2}V^2\sqrt{\chi r}} \int_{-\infty}^{\infty} \frac{exp(i\chi r(qcos\varphi + \sqrt{1+q^2}\,))}{(1+q^2)^{1/4}\sqrt{sin^2\theta + q^2}}\, dq$$

Let us consider the critical points of the integral (15). These points are, first, the stationary point $q = -ctg\varphi$ and, second, the two branching points $q = \pm isin\,\theta$ that are close to the real axis for small θ. If $ctg\varphi$ is not small, i.e., if $cos\varphi = 0(1)$, then the asymptotics of Λ is the sum of the contributions of these points. The contribution of the stationary point can be calculated standardly. The contribution of the points $\pm isin\,\theta$ coincides in the leading term of the asymptotics with the value of the model integral: $\int_{-\infty}^{\infty} \frac{exp(-i\chi rq\ cos\varphi)dq}{\sqrt{sin^2\theta + q^2}} = 2K_0(\chi rsin\,\theta cos\varphi) = 2K_0(\chi y)$, where $K_0(\tau)$ is the Bessel function of the imaginary argument [20,21]. As $\tau \to \infty$, $K_0(\tau)$ exponentially tends to zero, and as $\tau \to 0$, it increases logarithmically: $K_0(\tau) \approx \sqrt{\frac{\pi}{2\eta}}\exp(-\tau)(\tau \to \infty), K_0(\tau) \sim -ln\frac{\tau}{2} - C$, and $C = 0.5772...$ is the Euler constant [20,21]. Thus, if $sin\,\theta$ is not small (it suffices to put $sin\,\theta > (\chi r)^{-1/4}$), then the asymptotics of Λ is calculated by the stationary phase method [18]

$$\Lambda \approx \frac{-exp(i\chi rsin\varphi)}{2\pi V^2 \chi r\sqrt{sin^2\theta\ sin^2\varphi + cos^2\varphi}} \tag{16}$$

If $sin\theta$ is small, then $cos\varphi$ is not small ($cos\,\theta > (\chi r)^{-1/4}$) and one must take into account the contribution of the branching points $\pm isin\,\theta$

$$\Lambda \approx -\frac{exp(i\chi rsin\varphi)}{2\pi V^2 \chi r\sqrt{sin^2\varphi\, sin^2\theta + cos^2}} - \frac{exp(-i\pi/4 + i\chi r)K_0(\chi rsin\theta cos\varphi)}{\sqrt{2}\pi^{3/2}V^2\sqrt{\chi r}} \quad (17)$$

But if both $sin\theta$ and $cos\varphi$ are small, then the asymptotics (16) and (17) cannot be used. In this case, the stationary point turns out to be near the branching points and the model integral is the function that cannot be reduced to the known special functions [18-21]: $\Psi(\alpha,\beta) = \int\limits_{-\infty}^{\infty} \frac{exp(i(x-\alpha)^2)dx}{\sqrt{\beta^2 + x^2}}$

The asymptotics of Λ for small $cos\varphi$ and $sin\theta$ can be expressed in terms of the function $\Psi(\alpha,\beta)$

$$\Lambda \approx -\frac{exp(-i\pi/4 + i\chi r(1+sin^2\varphi)/2)}{(2\pi)^{3/2}V^2\sqrt{\chi r}}\Psi\left(-\sqrt{\frac{\chi r}{2}}cos\varphi, -\sqrt{\frac{\chi r}{2}}sin\theta\right) \quad (18)$$

4. Conclusion

Thus, we have considered the function $\Gamma(\xi,y,z)$ that has integral representation (4). All components of the IGW field generated by a moving point source of mass and dipole with the unit moment can be expressed in terms of the function Γ by explicit formulas. The asymptotic of Γ at large distances from the moving source of perturbations $\left(\chi = N/V, \xi = rcos\theta, y = rsin\theta cos\varphi, |z| = rsin\theta sin\varphi\right)$ for $|z| > (r/\chi)^{1/2}$, $|cos\theta| > (\chi r)^{-1/4}$ is expressed by formula (6). Near the traverse plane $\theta = \pi/2$, for $|z| > (r/\chi)^{1/2}$, $|cos\theta| < const < 1$, the asymptotics of Γ is determined by formula (7). Near the horizontal plane of the source motion, for $z < (r/\chi)^{1/2}$ and in front of the source, i.e., for $cos\theta < 0$ $(\pi/2 < \theta < \pi)$, the asymptotics of Γ is given by expression (12), and as $\chi z \to \infty$ and $cos\theta < 0$, this formula is asymptotically equivalent to (7). Finally, near the horizontal plane of the source motion $z < (r/\chi)^{1/2}$ and behind the source, i.e., for $cos\theta > 0$ and $sin\theta > (\chi r)^{-1/4}$, the far IGW fields are described by expressions (14)-(18). In the far region, the generated wave fields are relatively small with respect to the amplitude and, as a rule, are well described by linear equations. Therefore, it is not expedient to use direct numerical computations to study the far IGW propagation. As was shown in this paper, the analytic representations of far IGW fields are described by relatively simple analytic expressions. The initial and boundary conditions for specific sources of perturbation must be determined by the results of direct numerical modelling of the complete system of hydrodynamic equations or by purely evaluative semi empirical considerations, which permits adequately approximating the real nonlocal perturbation sources by a certain system of model sources. The obtained asymptotic solutions allow one efficiently to calculate the basic amplitude-phase characteristics of the generated far IGW fields in certain generation regimes and, in addition, qualitatively analyse the obtained solutions; this is important for the correct statement of more complicated mathematical models of wave dynamics of real natural stratified media (ocean, the Earth atmosphere). The model solutions permit obtaining further representations of wave fields with regard to the real variable and unstable behaviour of such media. The research was carried out in the framework of the Federal target program AAAA-A17-117021310375-7.

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