

Far Fields of Internal Gravity Waves at Arbitrary Movement Speeds of Source of Disturbances

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Abstract—The problem of a far field of internal gravity waves excited by a moving source of disturbances in a stratified medium is considered. The uniform asymptotics describing the amplitude–phase structure of the wave fields for the different conditions of a source movement are constructed. The wave pattern, including the position of a front, under subcritical conditions of source movement is determined by waves with the wave numbers restricted by a certain positive value from below, whereas under supercritical conditions the wave pattern is formed by the waves of all ranges and the front is determined only by long waves. In the first case, the phase portrait represents longitudinal and transverse waves decaying as a power function; in the second case, it is formed only of longitudinal waves decaying exponentially. In the first case, the uniform asymptotics of the far field consists of two terms, one of which is represented by the Airy function and the second is represented by its derivative; in the second case, the asymptotics has only one term that is expressed through the Airy function.

Keywords: internal gravity waves, stratified medium, far fields, uniform asymptotics

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One of the major mechanisms of excitement of internal gravity waves in the natural stratified media (an ocean or the atmosphere) is their generation by the sources of different physical nature disturbances, having both a natural (moving typhoon, floatation of ocean topographic inequalities, and downwind mountains) and an anthropogenic (marine technological structures, implosion of the region of turbulent mixing, and subsurface explosions) character [1–8]. In the linear approximation, the far wave fields can be studied using the asymptotics [3]. The plotted analytical structures make it possible to represent further wave fields, asymptotically taking into account the real heterogeneity and nonstationarity of the parameters of natural stratified media, including using the methods of computer mathematics [9].

We consider the problem of far fields of internal gravity waves that appear under the floatation of a point source of disturbances of thickness Q by a flow of stratified medium of thickness H . We assume that the source moves at the speed V horizontally along the x axis, the z axis is directed upward, the depth of the source occurrence is $-z_0$, and the regime of wave motions is steady. We have the following equation

$$V^2 \frac{\partial^2 \Delta \eta}{\partial x^2} + N^2(z) \Delta_2 \eta = QV \frac{d\delta(x)}{dx} \delta(y) \frac{d\delta(z - z_0)}{dz}, \quad (1)$$

in the linear statement and with respect to the Boussinesq approximation, for example, for the vertical

displacement of the isopycs (equal density lines)

$$\eta(x, y, z) \quad [1-4], \quad \text{where} \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2},$$

$$\Delta_2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad \text{and} \quad N^2(z) = -\frac{g}{\rho_0(z)} \frac{d\rho_0(z)}{dz}$$

is the Brunt–Väisälä frequency that is further assumed to be constant ($\rho_0(z)$ is the undisturbed density), $\delta(x)$ is the Dirac delta function. The function $\eta(x, y, z)$ is related to the vertical speed $w(x, y, z)$ by the ratio $w = V\partial\eta/\partial x$. The condition of a “solid cap” is used as boundary conditions

$$\eta = 0, \quad z = 0, -H. \quad (2)$$

Equation (1) and the boundary conditions (2) can be represented in the dimensionless coordinates

$$x^* = \frac{x\pi}{H}, \quad y^* = \frac{y\pi}{H}, \quad z^* = \frac{z\pi}{H}, \quad \text{and} \quad \eta^* = \frac{\eta VH}{\pi Q}$$

$$\frac{\partial^2 \Delta \eta}{\partial x^2} + \frac{1}{M^2} \Delta_2 \eta = \frac{d\delta(x)}{dx} \delta(y) \frac{d\delta(z - z_0)}{dz}, \quad (3)$$

$$\eta = 0, \quad z = 0, -\pi,$$

(the “*” sign is further omitted),

where $M = V/c$ and $c = NH/\pi$. In this statement of the problem, parameter M characterizes the ratio between the speed of the source of disturbances V and

the maximal value of the group speed c of the internal gravity waves [1–4].

The solution to the problem (3) is sought as the Fourier integral

$$\eta(x, y, z) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} dv \int_{-\infty}^{\infty} \varphi(\mu, v, z) \exp(-i(\mu x + vy)) d\mu. \quad (4)$$

Then, for function $\varphi(\mu, v, z)$, we obtain the following boundary problem $k^2 = \mu^2 + v^2$

$$\frac{\partial^2 \varphi}{\partial z^2} + k^2 (\mu^{-2} M^{-2} - 1) \varphi = \frac{1}{i\mu} \frac{d\delta(z - z_0)}{dz_0}, \quad (5)$$

$$\varphi = 0, \quad z = 0, -\pi.$$

The solution to problem (5) can be represented as a sum of the vertical (normal) modes [3, 4]

$$\varphi(\mu, v, z) = \sum_{n=1}^{\infty} \varphi_n(\mu, v, z) = \sum_{n=1}^{\infty} B_n(\mu, v) \sin nz, \quad (6)$$

i.e., as an eigen–function series for the homogeneous boundary problem that corresponds to Eq. (5). Substituting expansion (6) in (5) for $B_n(\mu, v)$, we obtain the following expression:

$$B_n(\mu, v) = \frac{2n \cos nz_0}{i\pi \mu k^2 (\mu^{-2} M^{-2} - 1) - n^2}.$$

Then the required solution to (4) is written as the sum of the modes

$$\eta(x, y, z) = \sum_{n=1}^{\infty} \eta_n(x, y, z),$$

$$\eta_n(x, y, z) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} dv \int_{-\infty+i\epsilon}^{\infty+i\epsilon} \frac{2n}{i\pi \mu} \times \frac{\exp(-i(\mu x + vy))}{k^2 (\mu^{-2} M^{-2} - 1) - n^2} d\mu \cos nz_0 \sin nz. \quad (7)$$

Equating the denominator in (7) to zero, we obtain a disperse ratio that binds the horizontal μ and the vertical v components of the wave vector k .

$$k^2 (\mu^{-2} M^{-2} - 1) = n^2. \quad (8)$$

We solve the biquadratic equation (8) relative to μ and obtain two real roots (the disperse curves)

$$\mu = \pm \mu_n(v), \quad \mu_n(v) = \sqrt{\frac{-(n^2 - m^2 + v^2) + \sqrt{(n^2 - m^2 + v^2)^2 + 4v^2 m^2}}{2}}, \quad (9)$$

where $m = 1/M$ and two imaginary roots

$$\mu = \pm i\lambda_n(v), \quad \lambda_n(v) = \sqrt{\frac{(n^2 - m^2 + v^2) + \sqrt{(n^2 - m^2 + v^2)^2 + 4v^2 m^2}}{2}}. \quad (10)$$

The contour of integration with respect to μ in the terms from (7) is displaced to the upper semiplane of the complex variable μ ($\epsilon > 0$). This is done to fulfill the condition of the study, when no wave motion in the upward direction along the flow is present [3, 4]. Closing the contour of integration with respect to μ to the upper semiplane (and taking the deductions into account at $\mu = \pm \mu_n(v)$ (9) and at $\mu = -i\lambda_n(v)$ (10)), we obtain

$$\eta_n(x, y, z) = \begin{cases} J_\lambda \frac{2n}{\pi} \sin nz \cos nz_0, & \text{at } x < 0 \\ (J_\mu^+ + J_\mu^- - J_\lambda) \frac{2n}{\pi} \sin nz \cos nz_0, & \text{at } x > 0, \end{cases} \quad (11)$$

$$J_\lambda = \frac{1}{\pi} \int_0^{\infty} A_\lambda(v) \exp(-\lambda_n(v)|x|) \cos vy dv,$$

$$J_\mu^\pm = \frac{1}{2\pi} \int_{-\infty}^{\infty} A_\mu(v) \cos(\mu_n(v)x \pm vy) dv, \quad (12)$$

$$A_\mu(v) = \frac{1}{2\mu_n^4(v)M^2 + v^2}, \quad A_\lambda(v) = \frac{1}{2\lambda_n^4(v)M^2 + v^2}.$$

Later on we consider only the most significant mode ($n = 1$), omitting subscript n , and the field of elevation η for this mode in two cases when $M < 1$ and $M > 1$.

The first case when $M < 1$ corresponds to the sub-critical conditions of movement of the source of disturbances. We first estimate the integral J_λ . At small v , the function $J(v)$ is expanded as $\lambda(v) = \frac{1}{\sqrt{1 - M^2}} v + O(v^3)$. Integrating by parts, we obtain $J_\lambda(x) = \frac{M^2}{2\pi\sqrt{1 - M^2}|x|} + O\left(\frac{1}{|x|^3}\right)$ at $|x| \rightarrow \infty$. The major contribution to the behavior of the field η in the far zone is made by the integrals J_μ^+ and J_μ^- . These integrals are equal to each other, since $\mu(v)$ is an even function; therefore, we consider further the integral J_μ^- . Due to the symmetry of the wave pattern relative to the x -axis, we assume $y > 0$ to be specific. Let us introduce the following designations: $\Phi = \mu(v)x - vy$, $\Phi = xS(v, \rho)$, $S(v, \rho) = \mu(v) - \rho v$, and $\rho = y/x$. Then the stationary points of the function $S(v, \rho)$ are determined from the equation

$$S'_v(v, \rho) = 0, \text{ or } \mu'(v) = \rho. \quad (13)$$

Adding the expression for the phase Φ to Eq. (13), we obtain the parameter family of equal phase lines (with the parameter v): $x(v) = \frac{\mu'(v)\Phi}{\mu(v) - \mu'(v)v}$. The disperse

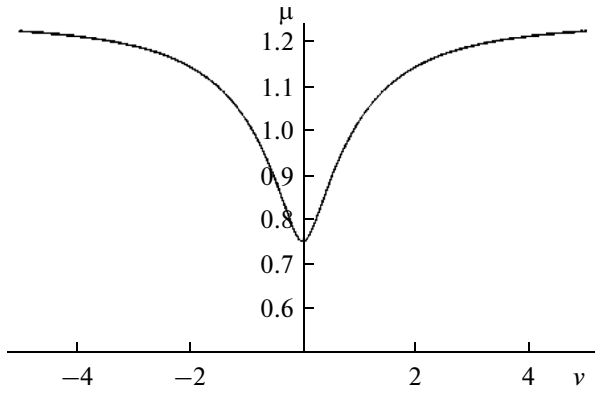


Fig. 1. Dispersion curve $\mu(v)$ ($M = 0.8$).

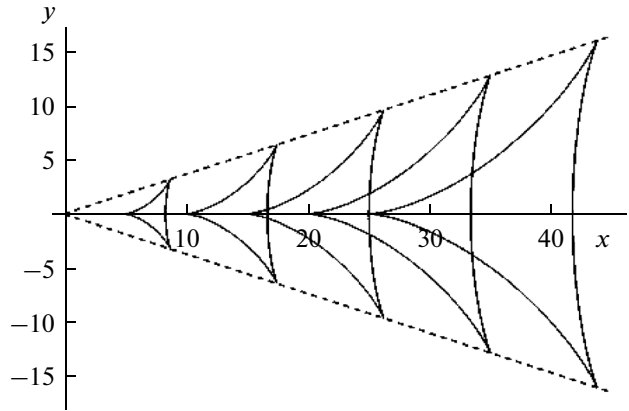


Fig. 2. Equal phase lines of the wave fields of elevations ($M = 0.8$).

curve $\mu(v)$ is depicted in Fig. 1 for the case when $M = 0.8$ at small v : $\mu(v) = \frac{\sqrt{1-M^2}}{M}v + O(v^3)$, at $v \rightarrow \infty$: $\mu(v) \rightarrow \frac{1}{M}$. The equal phase lines are plotted in Fig. 2 for $\Phi = 2\pi k$, $k = 1-5$ and they start at the points with the coordinates $(2\pi k M, 0)$. At the fixed ρ , we have two stationary points v_1 and v_2 , $v_2 > v_1$, when $v_2 > v_*$, and $v_1 < v_*$, where v_* is the inflection point $\mu(v)$ ($v_* \approx 0.32$). The stationary points $0 < v_1 < v_*$ correspond to the equal phase transverse lines (in the upper segment of the Figure), the stationary points $v_2 > v_*$ conform to the longitudinal lines. The wave front (the dashed lines) is described by the dependence: $y = \mu'(v_*)x$.

At big positive values of x , the integral J_μ^- can be calculated by the stationary phase method

$$J_\mu^- = J_1^- + J_2^-,$$

$$J_1^- = \frac{A(v_1(\rho))}{\sqrt{2\pi x S''_{vv}(v_1(\rho), \rho)}} \cos\left(xS(v_1(\rho), \rho) + \frac{\pi}{4}\right), \quad (14)$$

$$J_2^- = \frac{A(v_2(\rho))}{\sqrt{-2\pi x S''_{vv}(v_2(\rho), \rho)}} \cos\left(xS(v_2(\rho), \rho) - \frac{\pi}{4}\right).$$

The approximation (14) does not work in the vicinity of the front, where $\mu''(v) \rightarrow 0$, and the stationary points approach each other and the point v_* . Therefore, it is required to construct the uniform asymptotics to work in the front vicinity and to coincide with the approximation of the stationary phase (14) away from the front. The integral J_μ^- is classified as an integral with two coincident stationary points [10]. Next we replace the variables

$$S(v, \rho) = a + \sigma s - \frac{s^3}{3}, \quad (15)$$

and require that the point $v_1(\rho)$ corresponded to the stationary point $s_1 = -\sqrt{\sigma}$ and the point $v_2(\rho)$ corresponded to the point $s_2 = \sqrt{\sigma}$. Then, from (15) it follows that

$$a(\rho) = \frac{S(v_1(\rho), \rho) + S(v_2(\rho), \rho)}{2}, \quad (16)$$

$$\sigma(\rho) = \left(\frac{3}{4}(S(v_2(\rho), \rho) - S(v_1(\rho), \rho))\right)^{2/3}.$$

At the replacement of (15), the integral J_μ^- becomes

$$J_\mu^- = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(s) \cos\left(x\left(a + \sigma s - \frac{s^3}{3}\right)\right) ds, \quad (17)$$

$$G(s) = A(v(s)) \frac{dv}{ds}.$$

We represent the slowly changing function $G(s)$ in the form

$$G(s) = b_0 + b_1 s, \quad (18)$$

$$b_0(\rho) = \frac{G(\sqrt{\sigma}) + G(-\sqrt{\sigma})}{2}, \quad b_1(\rho) = \frac{G(\sqrt{\sigma}) - G(-\sqrt{\sigma})}{2\sqrt{\sigma}}. \quad (19)$$

The values $G(\pm\sqrt{\sigma})$ from the expression for dv/ds can be found by differentiating the equality (15) with respect to s two times. Then we have

$$G(\sqrt{\sigma}) = A(v_2(\rho)) \sqrt{\frac{-2\sqrt{\sigma(\rho)}}{S''_{vv}(v_2(\rho), \rho)}}, \quad (20)$$

$$G(-\sqrt{\sigma}) = A(v_1(\rho)) \sqrt{\frac{2\sqrt{\sigma(\rho)}}{S''_{vv}(v_1(\rho), \rho)}}.$$

Substituting the expressions (18) and (20) to (17), we write J_μ^- as the sum of two terms

$$J_\mu^- = I_1 + I_2, \quad (21)$$

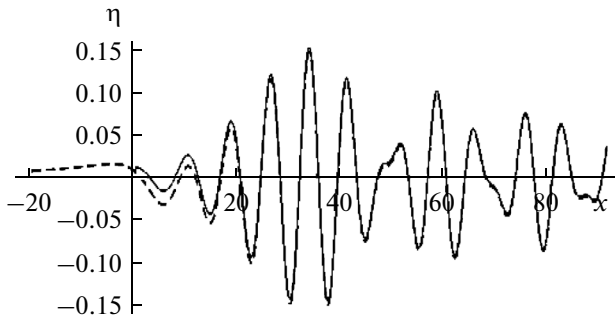


Fig. 3. Exact field of elevation (the dashed line) and uniform asymptotics (the solid line).

the first one is expressed through the Airy function

$Ai(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \cos\left(xt - \frac{t^3}{3}\right) dt$, and the second one is expressed through the derivative of the Airy function [10]

$$\begin{aligned} I_1 &= \frac{b_0(\rho)}{x^{1/3}} \cos(xa(\rho)) Ai(\sigma(\rho)x^{2/3}), \\ I_2 &= \frac{b_1(\rho)}{x^{2/3}} \sin(xa(\rho)) Ai'(\sigma(\rho)x^{2/3}), \end{aligned} \tag{22}$$

where $b_0(\rho)$ and $b_1(\rho)$ were calculated in (19) and $\sigma(\rho)$ and $a(\rho)$ were found in (16). At the slopes $\rho > \rho_*$, where $\rho_* = \mu'(v_*)$ ($\rho_* \approx 0.363$), the stationary points $v_1(\rho)$ and $v_2(\rho)$ become complex conjugate and expressions from (22) remain valid. The uniform asymptotics of (21) and (22) is regular in the vicinity of the wave front, where S'_v and S''_{vv} are equal to zero, with $b_0 = G(0)$ and $b_1 = G'(0)$. We can prove that the function G and its derivative are regular in zero by differentiating equality (15) in zero three and four times, respectively. The uniform asymptotics is matched with the nonuniform asymptotics (14) away from the front, which is shown by the substitution of the asymptotics of the Airy function and its derivative to the expression (22) for the Airy function and its derivative at big positive values of the argument: $Ai(x) \sim \frac{1}{x^{1/4}\sqrt{\pi}} \times \cos\left(\frac{2}{3}x^{3/2} - \frac{\pi}{4}\right)$ and $Ai'(x) \sim \frac{x^{1/4}}{\sqrt{\pi}} \cos\left(\frac{2}{3}x^{3/2} + \frac{\pi}{4}\right)$. We note that the both integrals in (21) have an equal order ($O(x^{-1/2})$) away from the wave front, and the major contribution is made by the integral I_1 in the front vicinity, which has the order $O(x^{-1/3})$, while the integral I_2 has the order $O(x^{-2/3})$.

Figure 3 shows the exact field of elevation η calculated by the formulas (11) and (12) (the dashed line) and the uniform asymptotics of the integral J_{μ}^- calculated from the formulas (21) and (22) (the solid line).

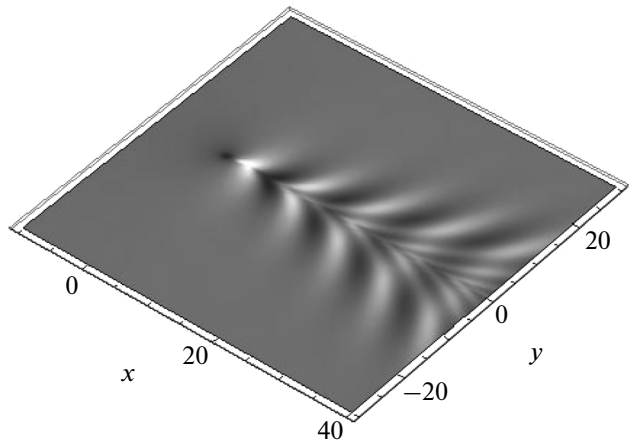


Fig. 4. Field of elevation of the internal gravity waves excited by the source of disturbance moving at a subcritical speed.

The calculations were performed for $M = 0.8$ at fixed $y = 13$, the multiplier $\frac{2}{\pi} \sin z \cos z_0$ from (11) was taken equal to one. Figure 4 depicts a 3D pattern of the exact field of elevation $\eta(x, y)$ calculated using the formulas (11) and (12) with the same values of M and the multiplier $\frac{2}{\pi} \sin z \cos z_0$ equal to one.

The second case when $M > 1$ corresponds to the supercritical regime of movement of the source of disturbances. The disperse curve $\lambda(v)$ in zero is expanded in $\lambda(v) = \frac{\sqrt{M^2 - 1}}{M} + O(v^2)$. This implies the exponential decay of the integral $J_{\lambda}(x)$ at $|x| \rightarrow \infty$: $J_{\lambda}(x) \sim O(|x|^{-1/2} \exp(-\sqrt{1 - M^{-2}}|x|))$. The disperse curve $\mu(v)$ is an uneven function. It is shown in Fig. 5 for $M = 1.8$. The integral J_{μ}^- makes the largest contribution to the behavior of the far wave field at $y > 0$, and the integral J_{μ}^+ makes the largest contribution to the behavior of the far field at $y < 0$. We further consider the integral J_{μ}^- to be specific. The function $S(v, \rho)$ is an uneven function of the variable v with a zero point of inflection. The stationary points calculated from (13) differ only in their sign in this case: $v_2(\rho) = v(\rho)$ and $v_1(\rho) = -v(\rho)$. The equal phase lines are shown in Fig. 6 for the case when $M = 1.8$, $\Phi = 2\pi k$, $k = 1, 2, 3, 4$. These are longitudinal lines (no transverse lines are present) originating at the points with the coordinates $2\pi k M, 0$ as in the case when $M < 1$. The wave front (the dashed lines) is described by the dependence $y = \mu'(0)x$, where $\mu'(0) = \frac{1}{\sqrt{M^2 - 1}}$. The uniform asymptotics of the integral J_{μ}^- looks much

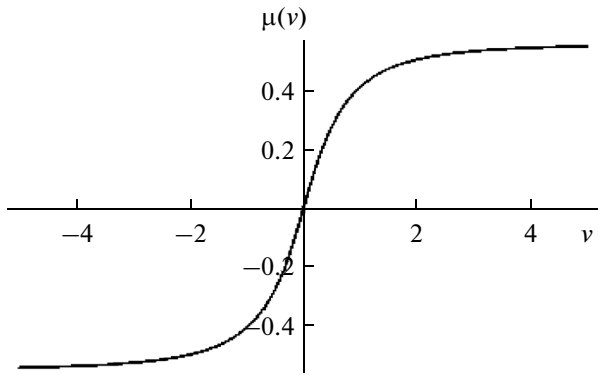


Fig. 5. Dispersion curve $\mu(v)$ ($M = 1.8$).

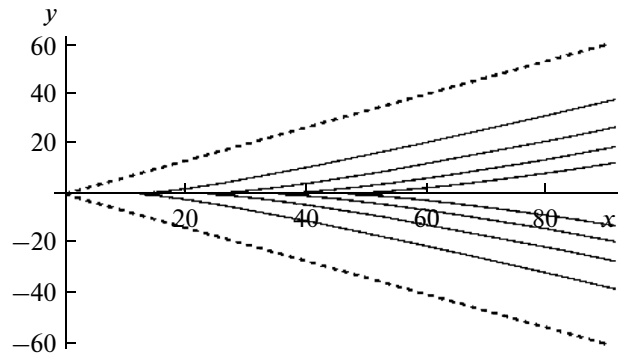


Fig. 6. Equal phase lines of the wave fields of elevations ($M = 1.8$).

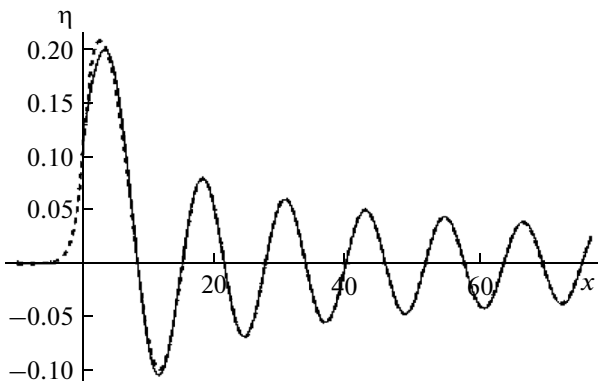


Fig. 7. Exact field of elevation (the dashed line) and uniform asymptotics (the solid line).

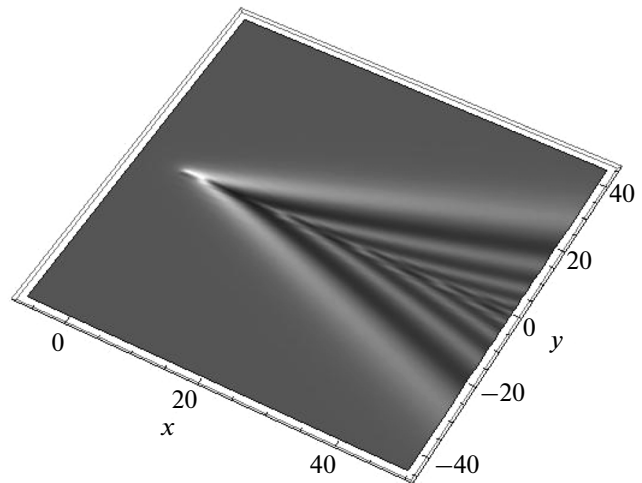


Fig. 8. Field of elevation of the internal gravity waves excited by the source of disturbance moving at a supercritical speed.

simpler than in the case when $M < 1$, since due to the symmetry of the stationary points and the uneven character of the function $S(v, \rho)$, the value a from (15) is equal to zero; therefore, the second term from (22) containing the derivative of the Airy function is absent [3, 4, 10].

$$J_{\mu}^{-} = \frac{A(v(\rho))}{x^{1/3}} \sqrt{\frac{2\left(\frac{3}{2}S(v(\rho), \rho)\right)^{1/3}}{-S''_{vv}(v(\rho), \rho)}} \times Ai\left(x^{2/3}\left(\frac{3}{2}S(v(\rho), \rho)\right)^{2/3}\right), \quad (23)$$

where the expressions for $A(v)$ and $S(v, \rho)$ are the same as for the case when $M < 1$. At big values of x , the uniform asymptotics (23) transforms to the nonuniform and is calculated by the stationary phase method

$$J_{\mu}^{-} = A(v(\rho)) \sqrt{\frac{2}{-\pi x S''_{vv}(v(\rho), \rho)}} \cos\left(xS(v(\rho), \rho) - \frac{\pi}{4}\right).$$

Figure 7 presents the exact field of elevation η calculated by the formulas (11) and (12) (the dashed line) and the uniform asymptotics determined according to

formula (23) (the solid line). The calculations were carried out for $M = 1.8$ at the fixed $y = 1$; the multiplier $\frac{2}{\pi} \sin z \cos z_0$ was taken equal to one. Figure 8 shows the 3D pattern of the exact field of elevation $\eta(x, y)$ obtained from formulas (11) and (12) with the same values of M and the multiplier $\frac{2}{\pi} \sin z \cos z_0$, which is equal to one.

We indicate the key quality differences of the wave pattern for the far fields of the internal gravity waves for the cases when $M < 1$ (the subcritical conditions of the movement of the source of disturbances) and when $M > 1$ (the supercritical conditions of the movement of the source of disturbances). In the first case, the wave pattern is determined by the values $k > k_0 = \sqrt{M^{-2} - 1}$ and the front is formed by the waves with $k_* = \sqrt{v_*^2 + \mu^2(v_*)}$; in the second case, the wave pat-

tern is formed by waves of an arbitrary length with $0 < k < \infty$ and the front is determined by the long waves with $k \rightarrow 0$. In the first case, the phase portrait consists of the longitudinal and transverse waves; in the second one, only from the longitudinal waves. In the first case, the wave field decreases in the upward direction along the flow as a power function (as x^{-1}) and, in the second case, it happens exponentially. In the first case, the uniform asymptotics of the far field consists of two terms, one of which is expressed through the Airy function, and the second term, through its derivative; in the second case, the asymptotics has only the term that is expressed via the Airy function.

CONCLUSIONS

The wave patterns similar to those we studied here may be recorded in remote sensing and observing the internal gravity waves excited by the different sources of disturbances, both in an ocean and the Earth's atmosphere [1, 5–8].

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