ISSN 0015-4628, Fluid Dynamics, 2015, Vol. 50, No. 6, pp. 741–747. © Pleiades Publishing, Ltd., 2015.
Original Russian Text © V.V. Bulatov, Yu.V. Vladimirov, 2015, published in Izvestiya Rossiiskoi Akademii Nauk, Mekhanika Zhidkosti i Gaza, 2015, Vol. 50, No. 6, pp. 26–34.

Internal Gravity Waves Excited by a Pulsating Source of Perturbations

V. V. Bulatov and Yu. V. Vladimirov

Ishlinsky Institute for Problems in Mechanics, Russian Academy of Sciences, pr. Vernadskogo 101, Moscow, 119526 Russia

e-mail: internalwave@mail.ru, vladimyura@yandex.ru

Received June 3, 2015

Abstract—The problem of constructing uniform asymptotics for the far fields of internal gravity waves generated by a pulsating localized source of perturbations in finite-depth stratified medium flow is considered. The solutions obtained describe the wave perturbations both inside and outside the wave fronts and can be expressed in terms of the Airy function and its derivatives. Numerically calculated wave patterns of the excited wave fields are presented.

Keywords: stratified medium, internal gravity waves, far fields, uniform asymptotics, wave front.

DOI: 10.1134/S0015462815060034

One of the main mechanisms of excitation of internal gravity waves in natural stratified media (ocean, atmosphere) is the wave generation by nonstationary sources of perturbations, different in physical nature, both of natural (moving typhoons, flow about ocean relief irregularities, leeward mountains, etc.) and anthropogenic (sea technological structures, collapse of turbulent mixing regions, underwater explosions, etc.) origin [1–5]. In the linear approximation, the far wave fields can be investigated, for example, using various asymptotics [3–9]. The analytical expressions constructed make it possible to obtain, using, for example, methods of computer mathematics, asymptotic representations of wave fields with account for the realistic inhomogeneity and non-stationarity of the parameters of natural stratified media [3–5].

The present study is aimed at constructing asymptotic solutions that can describe the far fields of the internal gravity waves excited by a pulsating localized source of perturbations in finite-depth stratified medium flow.

1. PROBLEM FORMULATION AND INTEGRAL FORMS OF THE SOLUTIONS

Consider the problem of far fields of the internal gravity waves that arise as a point source of power Q perturbations is flowed around by a stratified medium of thickness H. Assume that the source power is a harmonic function of time $Q = q\exp(i\omega t)$. The source moves at a velocity V in the horizontal direction along the x axis. The z axis is directed upward and the source location depth is z_0. The established regime of wave oscillations is considered.

In the linear formulation, in the Boussinesq approximation, we have the following equations, for example, for the vertical displacement of the isopycns $\eta(x, y, z)$ as curves of equal density with the same harmonic time dependence [3–5]:

$$\left(i\omega + V\frac{\partial}{\partial x}\right)^2(\Delta\eta) + N^2(z)(\Delta_2\eta) = Q\left(i\omega + V\frac{\partial}{\partial x}\right)\delta(x)\delta(y)\delta'(z - z_0), \tag{1.1}$$

$$\Delta = \Delta_2 + \frac{\partial^2}{\partial z^2}, \quad \Delta_2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},$$

where $N^2(z) = -\dfrac{g}{\rho_0(z)}\dfrac{d\rho_0(z)}{dz}$ is the Brent–Vaissala frequency assumed in what follows to be constant ($\rho_0(z)$ is the unperturbed density) and $\delta(x)$ the Dirac delta function. The function $\eta(x,\ y,\ z)$ is linked with the vertical velocity component $w(x,\ y,\ z)$ by the relation $w(x,\ y,\ z) = \left(i\omega + V\dfrac{\partial}{\partial x}\right)\eta(x,\ y,\ z)$ [3–5].

As boundary conditions we use the "rigid lid" condition

$$\eta = 0 \quad \text{at} \quad z = 0,\ -H. \tag{1.2}$$

In the dimensionless coordinates $x^* = x\pi/H,\ y^* = y\pi/H,\ z^* = z\pi/H,\ \eta^* = \eta H^2 V/q\pi^2,\ \omega^* = \omega/N,$ $t^* = tN$, Eq. (1.1) and boundary conditions (1.2) can be rewritten as follows (in what follows the superscript "*" is omitted):

$$\left(i\omega + M\frac{\partial}{\partial x}\right)^2(\Delta\eta) + (\Delta_2\eta) = \left(i\omega + M\frac{\partial}{\partial x}\right)\delta(x)\,\delta(y)\,\delta'(z - z_0), \tag{1.3}$$

$$\eta = 0 \quad \text{at} \quad z = 0,\ -\pi,$$

where $c = NH/\pi$ is the maximum value of the group velocity of the internal gravity waves in the stratified medium layer of thickness H [3–5] and $M = V/c$. In what follows we will consider the case of $M > 1$.

We will seek the solution of problem (1.3) in the form of the Fourier integral

$$\eta(x,\ y,\ z) = \frac{1}{4\pi^2}\int\limits_{-\infty}^{\infty} d\nu \int\limits_{-\infty}^{\infty} \varphi(\mu,\ \nu,\ z)\exp(-i(\mu x + \nu y))\,d\mu. \tag{1.4}$$

For the function $\varphi(\mu,\ \nu,\ z)$ we have the following boundary-value problem:

$$\frac{\partial^2\varphi}{\partial z^2} + k^2\left(\frac{1}{(\omega - \mu M)^2} - 1\right)\varphi = \frac{i}{(\omega - \mu M)}\frac{\partial\delta(z - z_0)}{\partial z_0}, \quad k^2 = \mu^2 + \nu^2, \tag{1.5}$$

$$\varphi = 0 \quad \text{at} \quad z = 0,\ -\pi.$$

The solution of problem (1.5) can be represented in the form of the sum of vertical (normal) modes $\varphi(\mu,\ \nu,\ z) = \sum\limits_{n=1}^{\infty}\varphi_n(\mu,\ \nu,\ z) = \sum\limits_{n=1}^{\infty} B_n(\mu,\nu)\cos nz_0\sin nz$, i.e., in the form of a series in the eigenfunctions of the homogeneous boundary-value problem (1.5), where

$$B_n(\mu,\ \nu) = \frac{2ni}{\pi(\omega - \mu M)}\frac{1}{k^2\Omega - n^2}, \qquad \Omega = \frac{1}{(\omega - \mu M)^2} - 1. \tag{1.6}$$

Finally, the solution of problem (1.4) can be represented in the form

$$\eta(x, y, z) = \sum_{n=1}^{\infty}\eta_n(x, y)\cos nz_0\sin nz, \quad \eta_n(x, y) = \frac{1}{4\pi^2}\int\limits_{-\infty}^{\infty} d\nu \int\limits_{-\infty}^{\infty} B_n(\mu,\ \nu)\exp(-i(\mu x + \nu y))\,d\mu.$$

Equating to zero the denominator $k^2\Omega - n^2$ from (1.6), we obtain the dispersion relation that links the horizontal μ and vertical ν components of the wave vector $\mathbf{k}$

$$k^2\left(\frac{1}{(\omega - \mu M)^2} - 1\right) = n^2, \quad n = 1,\ 2,\ldots \tag{1.7}$$

Now consider the case of $\omega < 1$ and, to be definite, the first wave mode ($n = 1$). The dispersion equation (1.7) has two real $\mu_1(\nu)$ and $\mu_2(\nu)$ and two complex conjugate $\lambda_1(\nu)$ and $\lambda_2(\nu)$ solutions. The problem of going around the poles $\mu_1(\nu)$ and $\mu_2(\nu)$ at the real axis of the integration variable μ is solved using the perturbation method. Replacing in (1.7) ω by $\omega - i\varepsilon$ ($\varepsilon > 0$), it can be shown that the imaginary parts of the perturbed solutions $\mu_1(\nu)$ and $\mu_2(\nu)$ are negative for all ν. Thus, in (1.6) the contour of integration with respect to the variable μ passes above the real axis.

We investigate the far fields of the internal gravity waves; therefore, when closing the contour of integration with respect to μ we only take into account the poles at the real axis, the contribution of the poles $\lambda_1(\nu)$ and $\lambda_2(\nu)$ to the total wave field being exponentially small at large $|x|$. Thus, we have the following expression for the function $\eta(x, y, t)$ at $|x| \gg 1$:

$$\eta(x, y, t) = \sum_{i=1}^{2} \frac{1}{2\pi} \int_{-\infty}^{\infty} A_i(\nu) \cos\Big(\mu_i(\nu)x - \nu y - \omega t\Big)\, d\nu = J_1 + J_2, \tag{1.8}$$

$$A_i(\nu) = \frac{M}{2} \frac{(\mu_i M - \omega)^2}{\mu_i \omega + M\nu^2 + \mu_i\big(\mu_i M - \omega\big)^3}.$$

2. CONSTRUCTING NONUNIFORM ASYMPTOTICS FOR THE SOLUTIONS

Consider the behavior of the integrals J_1 and J_2 from (1.8), which correspond to the dispersion curves $\mu_1(\nu)$ and $\mu_2(\nu)$ at large positive x. The dispersion relation (1.7) is a fourth-power algebraic equation of general form. Explicit formulas, like Ferrari's, are very cumbersome; nevertheless, it is the explicit formulas that are used by computational systems, like "Mathematica". This enables us to operate not only with the solutions $\mu_1(\nu)$ and $\mu_2(\nu)$ but also with their first- and second-order derivatives. In Fig. 1, the dispersion curves $\mu_1(\nu)$ and $\mu_2(\nu)$ are shown for $M = 1.8$ and $\omega = 0.52$. In what follows all calculations will be presented for the same parameter values.

First consider the integral J_1. Introduce the notation $\Phi_1 = \mu_1(\nu)x - \nu y - \omega t$. Then, using the phase stationarity condition in the form

$$\mu_1'(\nu) = y/x, \tag{2.1}$$

we obtain the family of constant-phase curves for various Φ_1 values with ν as a parameter

$$x = \frac{\Phi_1 + \omega t}{\mu_1(\nu) - \nu\mu_1'(\nu)}, \qquad y = \frac{\mu_1'(\nu)(\Phi_1 + \omega t)}{\mu_1(\nu) - \nu\mu_1'(\nu)}.$$

In Fig. 2, the equal-phase curves *1* are plotted for $t = 10$ and $\Phi_1 = 2\pi n$ ($n = 0, 1, 2, \ldots, 6.$) The right branch of the dispersion curve $\mu_1(\nu)$ ($\nu > 0$) corresponds to the upper part of the figure ($y > 0$). By virtue of symmetry, we will consider the upper region alone. In Fig. 1, the part of the dispersion curve from zero to point *A* corresponds to transverse waves, whereas the part from point *A* to infinity corresponds to longitudinal waves. Point *A* is an inflection point and corresponds to line *2* in Fig. 2, i.e., to the wave front within which the traveling wave described by the integral J_1 propagates. The equal-phase curves propagate from the origin of coordinates to infinity and along the ray have the phase velocity

$$C_f = \frac{\omega\sqrt{1 + \big(\mu_1'(\nu)\big)^2}}{\mu_1(\nu) - \nu\mu_1'(\nu)}.$$

Equations for the longitudinal waves in the neighborhood of the x axis can be obtained in the explicit form using the asymptotic behavior of the dispersion curve at large ν

$$\mu_1(\nu) = \frac{1 + \omega}{M} - \frac{1}{2M}\nu^{-2} + \ldots.$$

This equation has the form

$$y = \pm\frac{2\sqrt{2}(1 + \omega)^2}{3\sqrt{3(2\pi k + \omega t)}\,M^{3/2}}\left(x - \frac{(2\pi k + \omega t)M}{1 + \omega}\right)^{3/2}.$$

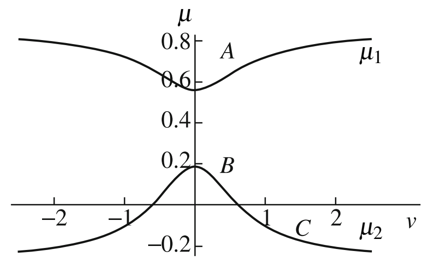

Fig. 1. Dispersion curves $\mu_1(\nu)$ (*1*) and $\mu_2(\nu)$ (*2*): A and B are points of inflection; C is root of the equation $\mu_2'(\nu) = -\mu/\nu$.

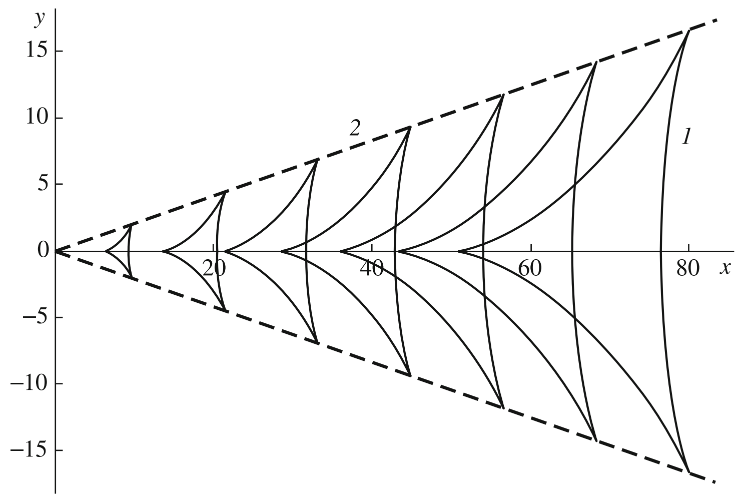

Fig. 2. Equal-phase curves for J_1: (*1*) equal-phase curves; (*2*) wave front.

Reproduce two other important characteristics of the wave fields, which are determined by the behavior of the integral J_1. The first of them is the wavelength, for example, the length of the transverse wave along the x axis $\lambda = 2\pi\mu_1(0) = 11.2$. The second is the angle of half-opening of the wave wedge $\Theta = \arctan\left(\mu_1'(\nu_*)\right) = 11.8^\circ$, where ν_* is the root of the equation $\mu_1''(\nu) = 0$ and the wave front itself has the form $y = Rx$, $R = \mu_1'(\nu_*)$.

In the stationary-phase approximation, the integral J_1 has the form [5, 10, 11]

$$J_1 = \frac{A_1(\nu_1)}{\sqrt{2\pi\mu_1''(\nu_1)x}} \cos\left(\mu_1(\nu_1)x - \nu_1 y - \omega t + \frac{\pi}{4}\right) + \frac{A_1(\nu_2)}{\sqrt{-2\pi\mu_1''(\nu_2)x}} \cos\left(\mu_1(\nu_2)x - \nu_2 y - \omega t - \frac{\pi}{4}\right), \tag{2.2}$$

where ν_1 and ν_2 are two roots of Eq. (2.1), ν_1 lying on the left of point A at the dispersion curve and ν_2 on the right. In (2.2), the first term describes transverse waves and the second longitudinal waves.

Now consider the integral J_2. Represent the phase Φ_2 in the form $\Phi_2 = -\mu_2(\nu)x - \nu y + \omega t$. The integral J_2 remains unchanged and at $y > 0$ the stationary points are positive. Then, the family of constant-phase curves at various Φ_2 with the parameter ν

$$x = \frac{\Phi_2 - \omega t}{-\mu_2(\nu) + \nu\mu_2'(\nu)}, \qquad y = \frac{\mu_2'(\nu)(\Phi_2 - \omega t)}{\mu_2(\nu) - \nu\mu_2'(\nu)}$$

is described by a more complicated pattern shown in Fig. 3. The ray equation has the form $\mu_2'(\nu) = -y/x$. In Fig. 1, the point of inflection B corresponds to the wave front (line *3* in Fig. 3), and point C, the root of the equation $\mu_2'(\nu) = -\mu/\nu$, corresponds to line *4* in Fig. 3. The part of the dispersion curve from zero to point B corresponds to transverse wave crests, the part from point B to point C to the longitudinal crests

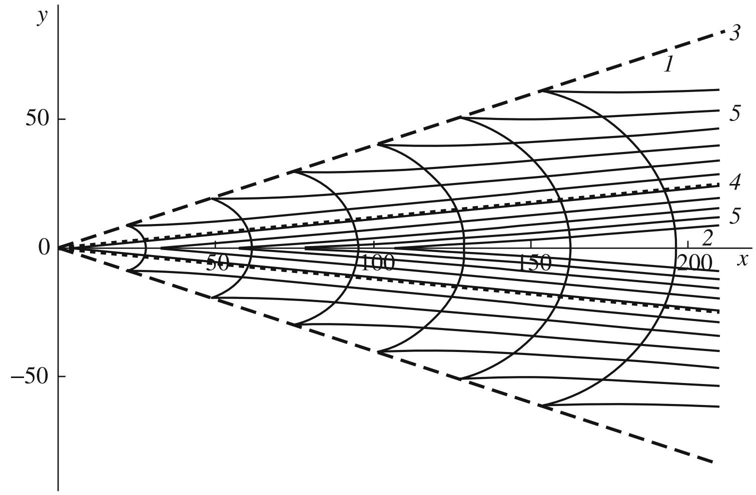

Fig. 3. Equal-phase curves for J_2: (*3*) wave front; (*5*) equal-phase curves; line (*4*) separates two regions (*1*) and (*2*) (equal-phase curves).

located between lines *3* and *4* in Fig. 3 (region 1), and the part to the right of point C to the longitudinal crests located between line *4* and the x axis (region 2). The crest phases which correspond to the location of the parameter ν at the dispersion curve from zero to point C have values $2\pi n$ ($n = 0, -1, -2, -3, -4$, and -5) and those which correspond to its location to the right of point C have values $2\pi n$ ($n = 0, 1, 2, 3, 4$, and 5) ($t = 10$). In region 1, the equal-phase curves propagate from the origin of coordinates to infinity and in region 2 from infinity to the origin of coordinates. In both cases, the group velocity vector is directed from the origin of coordinates toward infinity since this is the direction in which energy propagates. The length of the transverse wave along the x axis is equal to $\lambda = -2\pi/\mu_2(0) = 33.6$ and the angle of half-opening of the wave wedge to $\Theta = \arctan\left(\mu_2'(\nu_*)\right) = 21.7°$, where ν_* is the root of the equation $\mu_2''(\nu) = 0$. In the stationary-phase approximation, J_2 looks similar to (2.2).

3. CONSTRUCTING UNIFORM ASYMPTOTICS FOR THE SOLUTIONS

Consider uniform asymptotics for the integral J_1 (asymptotics for J_2 are constructed similarly). The stationary-phase approximation (2.2) does not work in the neighborhood of the wave front, where $\mu_1(\nu)'' \to 0$ and the stationary points merge. Therefore, we will represent the phase of the integral J_1 in the form $\Phi_1 = xS(\nu,\ r) - \omega t$, $S(\nu,\ r) = \mu_1(\nu) - r\nu$, $r = y/x$. In order to reduce the integral to the reference or canonical form, we will make the change of variables [5, 10, 11]

$$S(\nu,\ r) = a + \sigma s - s^3/3, \tag{3.1}$$

where the parameters a and σ can be unambiguously determined from the requirement of coincidence of the stationary points in (3.1), that is, the roots $\nu_1 = \nu_1(r)$ and $\nu_2 = \nu_2(r)$ of the equation $S'_\nu(\nu,\ r) = 0$ and $s_{1,2} = \pm\sqrt{\sigma}$: $a(r) = \left(S(\nu_1,\ r) + S(\nu_2,\ r)\right)/2$, $\sigma(r) = \left(3/4\left(S(\nu_1,\ r) - S(\nu_2,\ r)\right)\right)^{2/3}$. The uniform asymptotic approximation of the integral J_1 then has the form [10, 11]

$$J_1 = \frac{T_+(r)}{x^{1/3}}\mathrm{Ai}\,(x^{2/3}\sigma(r))\cos(a(r)x - \omega t) + \frac{T_-(r)}{x^{2/3}\sqrt{\sigma(r)}}\mathrm{Ai}'(x^{2/3}\sigma(r))\sin(a(r)x - \omega t), \tag{3.2}$$

$$T_\pm = \frac{1}{2}\left[\mathrm{A}_1(\nu_2)\sqrt{\frac{-2\sqrt{\sigma(r)}}{S''_{\nu\nu}(\nu_2,\ r)}} \pm A_1(\nu_1)\sqrt{\frac{2\sqrt{\sigma(r)}}{S''_{\nu\nu}(\nu_1,\ r)}}\right],$$

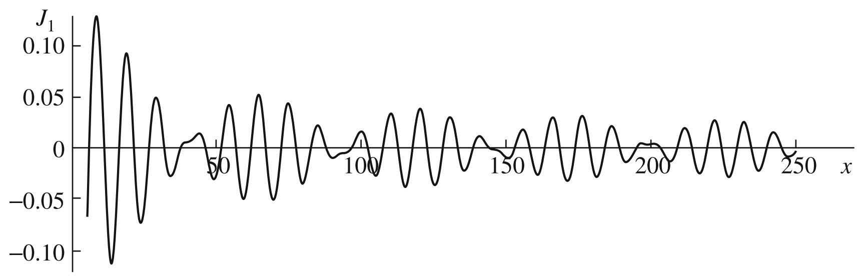

Fig. 4. Integral J_1 along the ray.

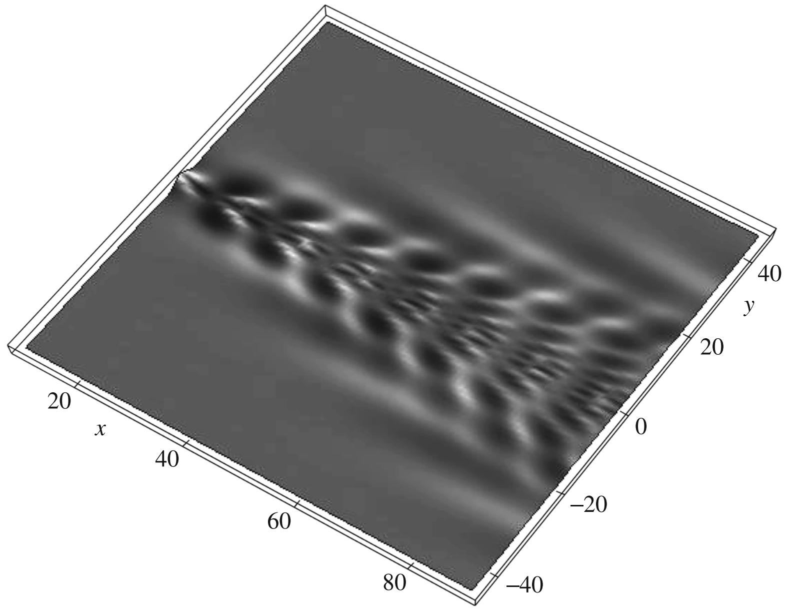

Fig. 5. Elevation of the internal gravity waves from a moving pulsating source of perturbations.

where the first term in (3.2) can be expressed in terms of the Airy function $\mathrm{Ai}\,(x) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} \cos\left(xt - \frac{t}{3}\right)^3 dt$ and the second in terms of the derivative of the Airy function.

The asymptotic approximation (3.2) is applicable in the neighborhood of the wave front, whereas far from the front it goes over into the nonuniform approximation (2.2), which can be verified by replacing the Airy function and its derivative by their asymptotics at large positive argument values

$$\mathrm{Ai}\,(x) \sim \frac{1}{\sqrt{\pi}x^{1/4}} \cos\left(\frac{2}{3}x^{3/2} - \frac{\pi}{4}\right), \qquad \mathrm{Ai}'(x) \sim \frac{x^{1/4}}{\sqrt{\pi}} \cos\left(\frac{2}{3}x^{3/2} + \frac{\pi}{4}\right).$$

Outside the wave wedge the uniform asymptotic approximation (3.2) is exponentially small since there are no real stationary points but a pair of complex conjugate points. Near the wave front the field is of the order $O(x^{-1/3})$, the first term in (3.2) predominating. Far from the front the asymptotic approximation (3.2) goes over into the stationary-phase approximation (2.2), both terms being of the order $O(x^{-1/2})$. Each of the two terms that form this approximation is a function rapidly oscillating in space, whose amplitude varies smoothly. The oscillation frequencies differ and are determined by the stationary points $\nu_1(r)$ and $\nu_2(r)$. In the uniform approximation, each term is also a rapidly oscillating function with slowly varying amplitude described by the Airy function and its derivative, the oscillation frequencies being the same and equal to the half-sum of eikonals at the points $\nu_1(r)$ and $\nu_2(r)$. The frequencies of the envelopes also coincide and are equal to the difference of eikonals at the same points. From this there follows that along the ray the uniform asymptotic approximation yields the wave pattern of pulsations. Figure 4 demonstrates the pattern of wave pulsations calculated by formula (3.2) for the integral J_1 at $r = 0.06$. The oscillation frequency is here approximately an order greater than the envelope frequency. In Fig. 5, the three-dimensional wave pattern of the elevation $\eta(x, y) = J_1 + J_2$ of the internal gravity waves from a pulsating source of perturbations is shown. In particular, the wave fronts corresponding to J_1 and J_2 can be seen.

Summary. The uniform asymptotic problem solutions obtained enabled us to describe the far fields of the internal gravity waves from a localized pulsating source of perturbations in finite-thickness stratified medium flow both outside and inside the corresponding wave fronts. It is shown that the far field asymptotics make it possible to efficiently calculate the main characteristics of the wave fields and to qualitatively analyze the solutions obtained. This opens wide opportunities for studying wave patterns as a whole, which is important for correctly constructing the mathematical models of gas dynamics, including for express-estimating in natural measurements of wave fields. Note that such wave patterns can be observed in the remote probing and observation of the internal gravity waves excited by various sources of perturbations in both the ocean and Earth's atmosphere [1, 5, 9].

The work was in part supported financially by the Russian Foundation for Basic Research (project No. 14-01-00466).

REFERENCES

1. J. Pedlosky, *Waves in the Ocean and Atmosphere: Introduction to Wave Dynamics* (Springer, Berlin & Heidelberg, 2010).
2. B.R. Sutherland, *Internal Gravity Waves* (Cambridge Univ. Press, Cambridge, 2010).
3. V.V. Bulatov and I.Yu. Vladimirov, *Dynamics of Nonharmonic Wave Packets in Stratified Media* (Nauka, Moscow, 2010) [in Russian].
4. V.V. Bulatov and Yu.V. Vladimirov, *Wave Dynamics of Stratified Mediums* (Nauka, Moscow, 2012).
5. V.V. Bulatov and Yu.V. Vladimirov, *Waves in Stratified Media* (Nauka, Moscow, 2015) [in Russian].
6. A. Zaitsev, "Waves Excited by a Subwater Source on the Surface of a Finite-Depth Flow," Dokl. Akad. Nauk SSSR **201** (5), 1070–1073 (1971).
7. A.M. Abdilghanie and P.J. Diamessis, "The Internal Gravity Wave Field Emitted by a Stably Stratified Turbulent Wake", J. Fluid Mech. **720**, 104–139 (2013).
8. T. Rees, K.G. Lamb, and F.J. Poulin, "Asymptotic Analysis of the Forced Internal Gravity Waves," SIAM J. Appl.Math. V. 72 (4), 1041–1060 (2012).
9. P.N. Svirkunov and M.V. Kalashnik, "Phase Plots for the Waves Generated by Localized Sources that Move Relative to a Rotating Stratified Medium (Traveling Hurricane, Orographic Obstacle)," Dokl. Ros. Akad. Nauk, Mechanics **447** (4), 396–400 (2012).
10. V.V. Bulatov, Yu.V. Vladimirov, and I.Yu. Vladimirov, "Uniform Asymptotics of the Far Fields of the Surface Disturbances Produced by a Source in a Heavy Infinite-Depth Fluid," Fluid Dynamics **49** (5), 655–661 (2014).
11. V.A. Borovikov, *Uniform Stationary Phase Method* (IEE electromagnetic waves, Ser. 40, London, 1994).