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Far Fields of Internal Gravity Waves Generated by a Perturbation Source in a Stratified Rotating Medium

V. V. Bulatov* **and Yu. V. Vladimirov****

Ishlinsky Institute for Problems in Mechanics, Russian Academy of Sciences,
pr. Vernadskogo 101, Moscow, 119526 Russia
*e-mail: *internalwave@mail.ru, **vladimyura@yandex.ru*

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Abstract—The problem of constructing uniform asymptotics for the far fields of internal gravity waves generated by a moving source of perturbations in flow of a finite-depth stratified rotating medium is considered. The solutions obtained describe the wave perturbations both inside and outside the wave fronts and can be expressed in terms of the Airy function and its derivatives. Numerically calculated wave patterns of the excited wave fields are presented.

Keywords: stratified rotating medium, internal gravity waves, far fields, uniform asymptotics, wave front.

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An important mechanism of excitement of internal gravity wave fields in ocean, Earth's atmosphere, and artificial stratified media is their generation by various perturbation sources whose nature can be natural (traveling typhoon, flow past roughness of the relief of the ocean bottom, leeward mountains) and anthropogenic (marine technological structures, collapse of the zone of turbulent mixing, underwater explosions). One of the basic factors that determines the characteristics of exited wave fields is rotation of the entire medium [1–8]. The available approaches to description of the wave patterns are based on representation of the wave fields by the Fourier integrals and an analysis their asymptotics using the stationary phase method ore on geometric construction of envelopes of the wave fronts within the framework of the kinematic theory of dispersing waves [1, 2, 6, 7]. Such wave patterns of the fields can be observed in distant sounding, observations, and measurements of internal gravity waves excited by various perturbation sources in natural (ocean and Earth's atmosphere) and artificial stratified rotating media [1, 2, 5, 6, 8].

As a rule, analytic representation can be formulated on the basis of the kinematic theory only for phase surfaces (curves) [8]. The aim of the present study is to construct asymptotic solutions describing the amplitude-phase characteristics for the far fields of internal gravity waves exited by a traveling perturbation source in a finite-depth stratified medium which rotates as a whole.

1. FORMULATION OF THE PROBLEM AND INTEGRAL FORMS OF THE SOLUTIONS

We will consider the problem of the far fields of internal gravity waves generated in flow past a point source of perturbations of power Q in a stratified medium of thickness H which rotates as a whole at a frequency Ω. The source moves with a velocity V in the horizontal direction along the x axis, the z axis is directed upward from the surface, the depth of position of the source is z_0. We will investigate the steady-state wave oscillation regime. Then in the linear formulation and with account for the Boussinesq approximation, for example, for a vertical displacement of isopycnics (equal-density lines) $\eta(x, y, z)$ we have the following equation [1–4]

$$V^2\frac{\partial^2}{\partial x^2}(\Delta\eta) + f^2\frac{\partial^2\eta}{\partial z^2} + N^2(z)(\Delta_2\eta) = Q\delta'(x)\delta(y)\delta'(z - z_0),$$
$$\Delta = \Delta_2 + \frac{\partial^2}{\partial z^2}, \qquad \Delta_2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x^2}, \tag{1.1}$$

where $N^2(z) = -\dfrac{g}{\rho_0(z)}\dfrac{d\rho_0(z)}{dz}$ is the Brunt-Väisälä frequency which is assumed to be constant in what follows, $\rho_0(z)$ is the unperturbed density, $\delta(x)$ is the Dirac delta-function, and $f = 2\Omega$ is the double frequency of rotation of the stratified medium.

The function $\eta(x, y, z)$ is connected with the vertical velocity component $w(x, y, z)$ by the relation $w(x, y, z) = V\dfrac{\partial}{\partial x^2}\eta(x, y, z)$ [3, 4]. As the boundary conditions, we will use the "rigid top" condition

$$\eta = 0 \quad \text{at} \quad z = 0,\ z = -H. \tag{1.2}$$

In the dimensionless coordinates $x^* = x\pi/H$, $y^* = y\pi/H$, $z^* = z\pi/H$, and $\eta^* = \eta H/\pi Q$ equation (1.1) and boundary conditions (1.2) can be written as follows (in what follows, we will omit the asterisk *):

$$\begin{gathered}\frac{\partial^2}{\partial x^2}(\Delta\eta) + \frac{\varepsilon^2}{M^2}\frac{\partial^2\eta}{\partial z^2} + \frac{1}{M^2}(\Delta_2\eta) = \delta'(x)\delta(y)\delta'(z - z_0),\\ \eta = 0 \quad \text{at} \quad z = 0,\ z = -\pi,\end{gathered} \tag{1.3}$$

where $c = NH/\pi$ is the maximum group velocity of the internal gravity waves in a layer of the stratified medium of thickness H [3, 4], $M = V/c$, and $\varepsilon = f/H$.

We will seek the solution of (1.3) in the form of the Fourier integral:

$$\eta(x, y, z) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty} d\nu \int_{-\infty}^{\infty} \varphi(\mu, \nu, z)\exp(-i(\mu x + \nu y))\,d\mu. \tag{1.4}$$

Substituting (1.4) in (1.3), we have the following boundary-value problem to determine the function $\varphi(\mu, \nu, z)$ ($k^2 = \mu^2 + \nu^2$):

$$\begin{gathered}\left(1 - \frac{\varepsilon^2}{\mu^2 M^2}\right)\frac{\partial^2\varphi(\mu, \nu, z)}{\partial z^2} + k^2\left(\frac{1}{(\mu M)^2} - 1\right)\varphi(\mu, \nu, z) = \frac{1}{i\mu}\frac{\partial\delta(z - z_0)}{\partial z_0},\\ \varphi(\mu, \nu, z) = 0 \quad \text{at} \quad z = 0,\ z = -\pi.\end{gathered} \tag{1.5}$$

The solution of the problem (1.5) can be represented in the form of the vertical (normal) modes: $\varphi(\mu, \nu, z) = \sum_{n=1}^{\infty}\varphi_n(\mu, \nu, z) = \sum_{n=1}^{\infty} B_n(\mu, \nu)\cos nz_0 \sin nz$, i.e., in the form of a series in the eigenfunctions of the homogeneous boundary-value problem (1.5), where $B_n(\mu, \nu) = 2n\mu M^2/(i\pi(k^2(1 - (\mu M)^2) - n^2((\mu M)^2 - \varepsilon^2)))$.

As a result, the solution of problem (1.3) takes the form:

$$\begin{gathered}\eta(x, y, z) = \sum_{n=1}^{\infty}\eta_n(x, y, z),\\ \eta_n(x, y, z) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty} d\nu \int_{-\infty}^{\infty} B_n(\mu, \nu)\exp(-i(\mu x + \nu y))\cos nz_0 \sin nz\,d\mu.\end{gathered} \tag{1.6}$$

Equating the denominator in (1.6) to zero, we obtain the dispersion relation which connects the horizontal and vertical component μ and ν of the wave vector k

$$\mu^2 M^2 = \frac{k^2 + n^2\varepsilon^2}{k^2 + n^2}, \quad n = 1, 2, \ldots \tag{1.7}$$

Solving the biquadratic equation (1.7) with respect for μ, we obtain two real roots (dispersion curves)

$$\mu = \pm\mu_n(\nu), \quad \mu_n(\nu) = \sqrt{\frac{(m^2 - \nu^2 - n^2) + \sqrt{(m^2 - \nu^2 - n^2)^2 + 4(\nu^2 + \varepsilon^2 n^2)m^2}}{2}} \tag{1.8}$$

and two imaginary roots

$$\mu = \pm i\lambda_n(\nu), \quad \lambda_n(\nu) = \sqrt{\frac{(n^2 - m^2 - \nu^2) + \sqrt{(n^2 - m^2 - \nu^2)^2 + 4(\nu^2 + \varepsilon^2 n^2)m^2}}{2}}, \tag{1.9}$$

where $m = 1/M$.

In terms of (1.6) the contour of integration with respect to μ is displaced toward the upper half-plane of the complex variable μ. The radiation condition, i.e., the absence of upstream wave propagation, is satisfied by this fact [3, 4]. Closing the contour of integration with respect to μ to the upper half-plane for $x < 0$ and to the lower half-plane for $x > 0$ and taking into account the residuals at $\mu = \pm\mu_n(\nu)$ (1.8) and $\mu = -i\lambda_n(\nu)$, we obtain

$$\eta_n(x,\ y,\ z) = \begin{cases} J_\lambda \dfrac{2n}{\pi} \sin nz \cos nz_0, & x < 0, \\ \left(J_\mu^+ + J_\mu^- - J_\lambda\right) \dfrac{2n}{\pi} \sin nz \cos nz_0, & x > 0, \end{cases}$$

$$J_\lambda = \frac{1}{\pi}\int\limits_0^\infty B_n(\nu)\exp(-\lambda_n(\nu)|x|)\cos \nu y\, d\nu, \quad J_\mu^\pm = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} A_n(\nu)\cos(\mu_n(\nu)x \pm \nu y)\, d\nu,$$

$$A_n(\nu) = \frac{1}{2}\frac{\mu_n^2(\nu)M^2}{\mu_n^4(\nu) + n^2\varepsilon^2 + \nu^2}, \quad B_n(\nu) = \frac{1}{2}\frac{\lambda_n^2(\nu)M^2}{\lambda_n^4(\nu) + n^2\varepsilon^2 + \nu^2}.$$

2. CONSTRUCTION OF ASYMPTOTICS OF THE SOLUTIONS

In what follows, to be specific, we will consider the first wave mode $n = 1$ (omitting its number) and the case $M > 1$ since in the case of $M < 1$ the dispersion curves $\mu(\nu)$ considered with and without rotation of stratified medium are not radically different.

Initially, we will estimate the integral J_λ. Since $\lambda(\nu) > 1$ for all ν, then J_λ is exponentially small for large $|x|$. Therefore, as $x \to \infty$, the behavior of the far fields is determined by the integrals J_μ^+ and J_μ^- which are identical one another since $\mu(\nu)$ is the even function.

We will estimate asymptotics of the integral J_μ^-. For the sake of definiteness, by virtue of symmetry of the wave pattern about the Ox axis we will assume that $y > 0$. We introduce $\Phi(\nu,\ \rho) = xS(\nu,\ \rho)$, $S(\nu,\ \rho) = \mu(\nu) - \rho\nu$, $\rho = y/x$. Then the stationary points of the function $S(\nu,\ \rho)$ can be determined from the equation

$$\frac{\partial\Phi(\nu,\ \rho)}{\partial\nu} = 0 \quad \text{or} \quad \mu'(\nu) = \rho. \tag{2.1}$$

Then, adding to (2.1) the expression for the phase Φ, we can obtain the parametric family of the equal-phase lines (with the parameter ν)

$$x(\nu) = \frac{\Phi}{\mu(\nu) - \mu'(\nu)\nu}, \quad y(\nu) = \frac{\mu'(\nu)\Phi}{\mu(\nu) - \mu'(\nu)\nu}.$$

For the fixed value of Φ we have a particular equal-phase line. In Fig. 1 we have reproduced the dispersion curves $\mu(\nu)$ for $M = 1.3$ and $\varepsilon = 0$, 0.11, and 0.3 (curves *1–3*, respectively). All the dispersion curves have a horizontal asymptote $\mu = 1/M$. The half-angle of the wave wedge α can be calculated from the formulas $\alpha = \tan^{-1}(\mu'(\nu_*))$, where ν_* is a root of the equation $\mu''(\nu) = 0$.

In Fig. 2 we have reproduced the graph of the function $\alpha(\varepsilon)$, in this case $\alpha(0) = \tan^{-1}(1/\sqrt{M^2 - 1})$. In Figs. 3 and 4 we have reproduced the equal-phase lines for $M = 1.3$ and $\varepsilon = 0.3$ and 0.11, respectively (curves *1*). For each value of ρ the solution of Eq. (2.1) gives two stationary points $\nu_1(\rho)$ and $\nu_2(\rho)$ so that $\nu_1 < \nu < \nu_2$. The longitudinal and transverse equal-phase lines correspond to the stationary points $\nu_* < \nu_2$ and $\nu_1 < \nu_*$, respectively. The stationary points merge on the wave front (see curve *2* in Figs. 3 and 4), the front being described by the equation $y = \mu'(\nu_*)x$. The distances between the longitudinal waves along

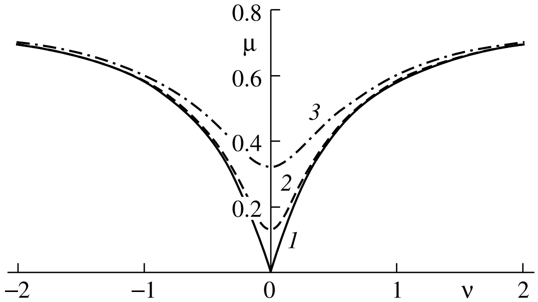

Fig. 1.

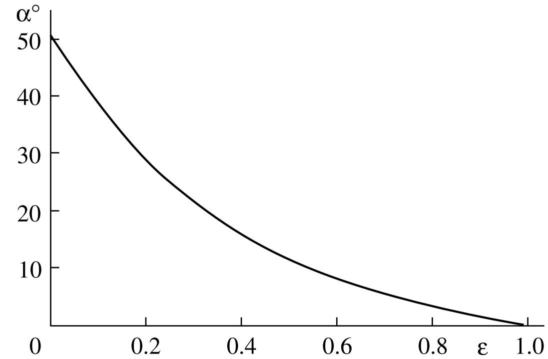

Fig. 2.

Fig. 1. Dispersion curves $\mu(\nu)$ for various values of the rotation parameter $\varepsilon = 0$: curves *1–3* correspond to $\varepsilon = 0$, 0.11, and 0.3, respectively.

Fig. 2. Half-angle of the wave wedge as a function of the rotation parameter ε.

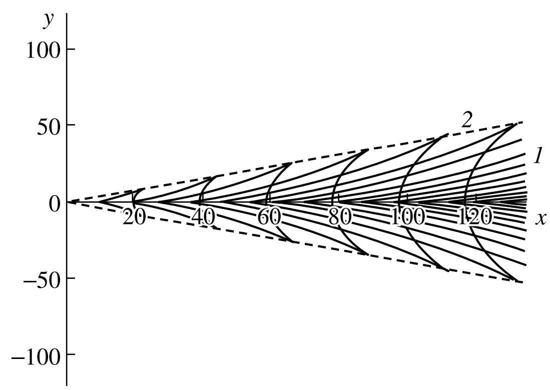

Fig. 3.

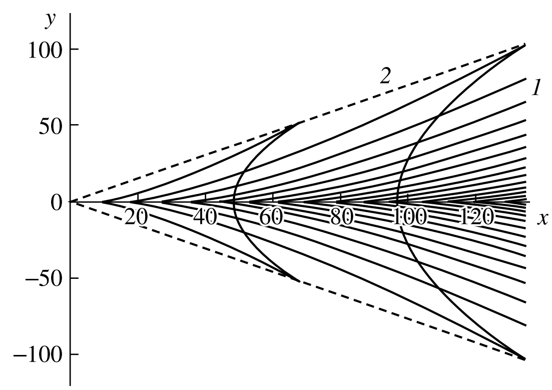

Fig. 4.

Fig. 3. Lines of equal phase (curves *1*) and the wave front (curve *2*) for $\varepsilon = 0.3$ ($\alpha = 21.4°$).

Fig. 4. Lines of equal phase (curves *1*) and the wave front (curve *2*) for $\varepsilon = 0.11$ ($\alpha = 37.6°$).

the Ox axis are equal to $2\pi M$ and between the transverse waves $2\pi/\mu(0)$. As the frequency of rotation of stratified medium ε decreases, the distance between transverse waves increases and in the limit $\varepsilon = 0$ there are no transverse waves, while the distance between longitudinal waves remains the same [4, 9]. At low ε the distance between transverse waves can be calculated using the approximate formula $2\pi\sqrt{M^2-1}/\varepsilon$ and, for example, at $\varepsilon = 0.11$ the relative error is not greater than 2%. For large positive x the nonuniform asymptotics of the integral J_μ^- can be calculated by means of the steady-state phase method [3, 4, 9]:

$$J_\mu^- = \frac{A(\nu_1(\rho))\cos(xS(\nu_1(\rho),\,\rho)+\pi/4)}{\sqrt{2\pi x S''_{\nu\nu}(\nu_1(\rho),\,\rho)}} + \frac{A(\nu_2(\rho))\cos(xS(\nu_2(\rho),\,\rho)-\pi/4)}{\sqrt{-2\pi x S''_{\nu\nu}(\nu_2(\rho),\,\rho)}}. \tag{2.2}$$

The nonuniform asymptotics (2.2) cannot be used in the neighborhood of the wave front, where the stationary points merge and $S''_{\nu\nu}(\nu(\rho),\,\rho) \to 0$. The uniform asymptotics of the integral J_μ^- which can be used both on the front and inside and outside the wave wedge has the form [3, 4, 9]:

$$\begin{gathered} J_\mu^- = \frac{b_0(\rho)}{x^{1/3}}\cos(xa(\rho))Ai\big(\sigma(\rho)x^{2/3}\big) + \frac{b_1(\rho)}{x^{1/3}}\sin(xa(\rho))Ai'\big(\sigma(\rho)x^{2/3}\big), \\ a(\rho) = \frac{S(\nu_1(\rho),\,\rho)+S(\nu_2(\rho),\,\rho)}{2}, \quad \sigma(\rho) = \left(\frac{3}{4}\big(S(\nu_2(\rho),\,\rho)-S(\nu_1(\rho),\,\rho)\big)\right)^{2/3}, \\ b_0(\rho) = \frac{G(\sqrt{\sigma})+G(\sqrt{-\sigma})}{2}, \qquad b_1(\rho) = \frac{G(\sqrt{\sigma})-G(\sqrt{-\sigma})}{2\sqrt{\sigma}}, \end{gathered} \tag{2.3}$$

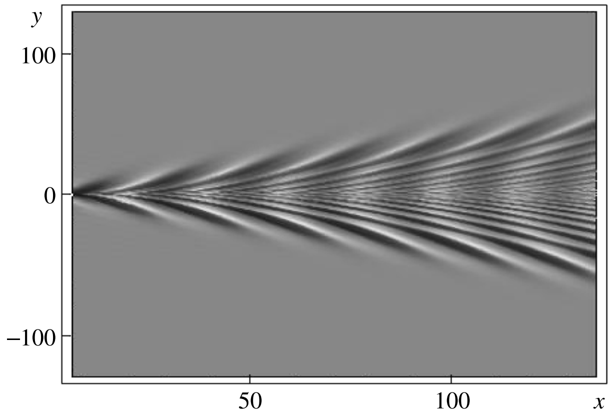

Fig. 5.

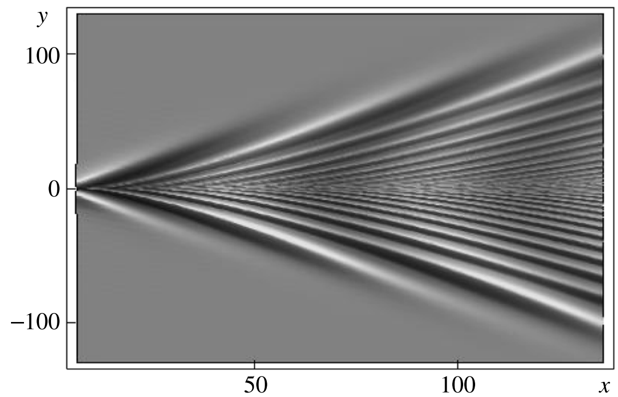

Fig. 6.

Fig. 5. Elevation of internal gravity waves generated by a source in a stratified rotating medium ($\varepsilon = 0.3$ and $\alpha = 21.4°$).

Fig. 6. Elevation of internal gravity waves generated by a source in a stratified rotating medium ($\varepsilon = 0.11$ and $\alpha = 37.6°$).

$$G(\sqrt{\sigma}) = A(\nu_2(\rho))\sqrt{\frac{-2\sqrt{\sigma(\rho)}}{S''_{\nu\nu}(\nu_2(\rho),\rho)}}, \quad G(-\sqrt{\sigma}) = A(\nu_1(\rho))\sqrt{\frac{2\sqrt{\sigma(\rho)}}{S''_{\nu\nu}(\nu_1(\rho),\rho)}},$$

where $Ai(x) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} \cos\left(xt - \frac{t}{3}\right)^3 dt$ is the Airy function.

The uniform asymptotics (2.3) expressed in terms of the Airy function and its derivative are regular in the neighborhood of the wave front, where S'_ν and $S''_{\nu\nu}$ tend to zero. The uniform asymptotics (2.3) coincide with the nonuniform asymptotics (2.2). We can check this fact by substituting the asymptotics of the Airy function and its derivative for large positive values of the argument in place of the Airy function and its derivative themselves:

$$Ai(x) \sim \frac{1}{x^{1/4}\sqrt{\pi}}\cos\left(\frac{2}{3}x^{3/2} - \frac{\pi}{4}\right), \quad Ai'(x) \sim \frac{x^{1/4}}{\sqrt{\pi}}\cos\left(\frac{2}{3}x^{3/2} + \frac{\pi}{4}\right).$$

Each of the terms in (2.3) consists of a rapidly oscillating (trigonometric) multiplier and a slowly varying amplitude (the Airy function and its derivative). In Figs. 5 and 6 we have reproduced the results of calculations of the three-dimensional pattern of the elevation field of internal gravity waves calculated from (2.3) for $\varepsilon = 0.3$ and 0.11, respectively. We can see that taking rotation of the stratified medium as a whole into account leads to a appreciable complication of both amplitude and phase characteristics of the generated far wave fields, namely, to appearance of not only the longitudinal but also the transverse wave packets which are absent in media without rotation [3, 4].

Summary. Uniform asymptotic solutions constructed in the study make it possible to describe the amplitude-phase characteristics of far fields of internal gravity waves generated by a local perturbation source moving in flow of a finite-depth stratified medium, which rotates as a whole, both outside and inside the corresponding wave fronts.

The asymptotics of far fields obtained make it possible not only to calculate efficiently the basic characteristics of wave fields but also to analyze qualitatively the solutions obtained.

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