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INTERNAL GRAVITY WAVES GENERATED BY AN OSCILLATING SOURCE OF PERTURBATIONS MOVING WITH SUBCRITICAL VELOCITY

V. V. Bulatov and Yu. V. Vladimirov

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Abstract: This paper considers the problem of constructing far-field asymptotics of internal gravity waves generated by an oscillating local source of perturbations moving in a stratified flow of finite depth. The velocity of the perturbation source does not exceed the maximum group velocity of an individual wave mode. The wave pattern consists of waves of two types: annular and wedge-shaped. Solutions expressed in terms of the Hankel function are obtained for the asymptotics of annular waves. The asymptotics of wedge-shaped waves are expressed in terms of the Airy function and its derivative.

Keywords: stratified fluid, internal gravity waves, far fields, uniform asymptotics, wave front.

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INTRODUCTION

The fields of internal gravity waves in natural (ocean, Earth atmosphere) and artificial stratified fluids are generated by perturbation sources of different origin: natural (moving typhoon, flow around the irregularities of the ocean bottom and land surface) and anthropogenic (marine technological structures, collapse of turbulent mixing regions, underwater explosions) [1–6]. Existing approaches to the description of the wave pattern are based on the representation of wave fields in the form of Fourier integrals and the analysis of their asymptotics by the method of stationary phase or on the construction of envelopes of wave fronts with the use of the kinematic theory of dispersive waves [4, 5, 7, 8]. As a rule, kinematic theory makes it possible to formulate an analytical representation only for phase surfaces (lines) [9].

The aim of this paper is to construct asymptotic solutions describing the amplitude–phase characteristics of the far fields of internal gravity waves generated by oscillating an source of perturbations moving in a stratified fluid of finite depth. Similar wave motions resulting from a superposition of translational and vibrational motion of the body, in a stratified fluid were considered in [10–12].

FORMULATION OF THE PROBLEM AND INTEGRAL FORMS OF SOLUTIONS

In this paper, we consider the problem of the far fields of internal gravity waves arising in a stratified flow of depth H past a point source of perturbations with power Q. It is assumed that the time dependence of the power of the source is harmonic: $Q = q\exp(i\omega t)$. The source moves with velocity V along the horizontal x axis, the axis z is directed upward, and the depth of the source is $-z_0$. The steady-state regime of wave oscillations is considered. In a linear formulation with the Boussinesq approximation for the vertical displacement of the density contours with the same time dependence, we write the equation [3–5, 13]

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$$\left(i\omega + V\,\frac{\partial}{\partial x}\right)^2 \Delta\eta + N^2(z)\,\Delta_2\eta = Q\left(i\omega + V\,\frac{\partial}{\partial x}\right)\delta(x)\delta(y)\,\frac{\partial\delta(z-z_0)}{\partial z_0},$$

$$\Delta = \Delta_2 + \frac{\partial^2}{\partial z^2}, \qquad \Delta_2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \tag{1}$$

where $N^2(z)$ is the Brunt–Väisälä frequency, which is assumed to be constant:

$$N^2(z) = -\frac{g}{\rho_0(z)}\,\frac{d\rho_0(z)}{dz},$$

$\rho_0(z)$ is the unperturbed density of the fluid and $\delta(x)$ is the Dirac delta function. The function $\eta(x,y,z)$ is linked to the vertical velocity component $w(x,y,z)$ by the relation [3–5, 13]

$$w(x,y,z) = \left(i\omega + V\,\frac{\partial}{\partial x}\right)\eta(x,y,z).$$

The boundary conditions are given by the rigid lid condition

$$z = 0, \quad z = -H: \qquad \eta = 0. \tag{2}$$

In the dimensionless variables

$$x^* = x\pi/H, \quad y^* = y\pi/H, \quad z^* = z\pi/H, \quad \eta^* = \eta H^2 V/(q\pi^2), \quad \omega^* = \omega/N, \quad t^* = tN,$$

Eq. (1) and boundary conditions (2) are written as follows (the superscript asterisk is omitted):

$$\left(i\omega + \mathrm{M}\,\frac{\partial}{\partial x}\right)^2 \Delta\eta + \Delta_2\eta = \left(i\omega + \mathrm{M}\,\frac{\partial}{\partial x}\right)\delta(x)\delta(y)\delta'(z-z_0),$$

$$z = 0, \quad z = -\pi: \qquad \eta = 0. \tag{3}$$

Here $\mathrm{M} = V/c$ is the Mach number; $c = NH/\pi$ is the maximum group velocity of internal gravity waves in a stratified fluid layer of depth H [3–5]. In a previous study [13], we have considered the case $\mathrm{M} > 1$ and have shown that far from an oscillating source of perturbations, the generated fields form a system of wedge-shaped waves enclosed in the corresponding wave fronts. In this paper, we consider the case $\mathrm{M} < 1$.

As in [13], the solution of problem (3) is sought in the form of the Fourier integral

$$\eta(x,y,z) = \frac{1}{4\pi^2}\int\limits_{-\infty}^{\infty} d\nu \int\limits_{-\infty}^{\infty} \varphi(\mu,\nu,z)\,\mathrm{e}^{-i(\mu x+\nu y)}\,d\mu.$$

Then, we have the following boundary-value problem for the function $\varphi(\mu,\nu,z)$:

$$\frac{\partial^2\varphi}{\partial z^2} + k^2\left(\frac{1}{(\omega-\mu\mathrm{M})^2} - 1\right)\varphi = \frac{i}{\omega-\mu\mathrm{M}}\,\frac{\partial\delta(z-z_0)}{\partial z_0},$$

$$z = 0, \quad z = -\pi: \qquad \varphi = 0 \tag{4}$$

($k^2 = \mu^2 + \nu^2$).

The solution of problem (4) is represented as the sum of vertical modes:

$$\varphi(\mu,\nu,z) = \sum_{n=1}^{\infty}\varphi_n(\mu,\nu,z) = \sum_{n=1}^{\infty} B_n(\mu,\nu)\cos(nz_0)\sin(nz),$$

$$B_n(\mu,\nu) = \frac{2ni}{\pi(\omega-\mu\mathrm{M})}\,\frac{1}{k^2((\omega-\mu\mathrm{M})^{-2}-1)-n^2},$$

i.e., in the form of a series in eigenfunctions of the homogeneous boundary-value problem (4). As a result, the solution of problem (3) has the form

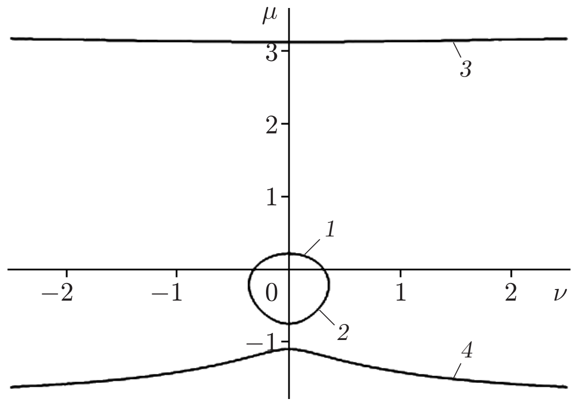

Fig. 1. Dispersion curves of the first type (*1* and *2*) and second type (*3* and *4*): $\mu_2(\nu)$ (*1*), $\mu_3(\nu)$ (*2*), $\mu_1(\nu)$ (*3*), and $\mu_4(\nu)$ (*4*).

$$\eta(x, y, z) = \sum_{n=1}^{\infty} \eta_n(x, y) \cos(nz_0) \sin(nz),$$

$$\eta_n(x, y) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} d\nu \int_{-\infty}^{\infty} B_n(\mu, \nu)\, \mathrm{e}^{-i(\mu x + \nu y)}\, d\mu. \tag{5}$$

Equating the denominator in (5) to zero, we obtain the dispersion relation connecting the horizontal component μ and vertical component ν of the wave vector k:

$$k^2 \Big(\frac{1}{(\omega - \mu \mathrm{M})^2} - 1 \Big) = n^2, \qquad n = 1, 2, \dots . \tag{6}$$

Consider the first wave mode ($n = 1$) for the case $\omega < 1$. For $\mathrm{M} < 1$, the dispersion equation (6) has two to four real roots. Figure 1 shows dispersion curves of two types: the curve of the first type is a closed curve [in Fig. 1, it consists of two parts $\mu_2(\nu)$ and $\mu_3(\nu)$], and the curves of the second type are two non-closed curves $\mu_1(\nu)$ and $\mu_4(\nu)$. Here and below, all the results of numerical calculations are given for $\mathrm{M} = 0.4$ and $\omega = 0.3$. The problem of going around the poles $\mu_i(\nu)$ ($i = 1, 2, 3, 4$) on the real axis during integration over the variable μ is solved by the perturbation method. Replacing ω in (6) by $\omega - i\varepsilon$ ($\varepsilon > 0$), we obtain the perturbed solutions $\mu_i(\nu) - i\varepsilon r_i(\nu)$, where

$$r_i(\nu) = \Big(\mathrm{M} - \frac{\mu_i(\nu)}{(\mu_i(\nu)\mathrm{M} - \omega)(\mu_i^2(\nu) + \nu^2 + 1)^2} \Big)^{-1}.$$

We can show that $r_i(\nu) > 0$ ($i = 1, 2, 4$) and $r_3(\nu) < 0$ for all ν, so that the poles $\mu_i(\nu)$ ($i = 1, 2, 4$) are bypassed from above, and the pole $\mu_3(\nu)$ from below. During integration over the variable μ, the integration contour shown in Fig. 2 (the poles $\mu_i(\nu)$ ($i = 1, 2, 3, 4$) correspond to the value $\nu = 0$) is closed downward at $x > 0$ and upward at $x < 0$. Then, the exact solution (taking into account the harmonic dependence of η on time) has the form

$$\eta(x, y, t) = \begin{cases} J_1 + J_2 + J_4, & x > 0, \\ -J_3, & x < 0, \end{cases}$$

where

$$J_j = -\frac{1}{2\pi} \int_{-\infty}^{\infty} f_j(\nu)\, \mathrm{e}^{-i(\mu_j(\nu)x + \nu y - \omega t)}\, d\nu,$$

$$f_j(\nu) = \frac{\mathrm{M}}{2} \frac{(\mu_i \mathrm{M} - \omega)^2}{\mu_j(\nu)\omega + \mathrm{M}\nu^2 + \mu_j(\nu)(\mu_j(\nu)\mathrm{M} - \omega)^3}, \qquad j = 1, 2, 3, 4.$$

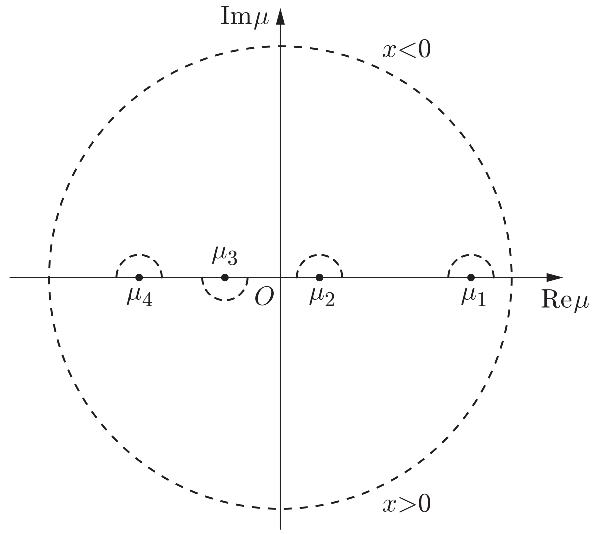

Fig. 2. Integration contour and the paths around the poles.

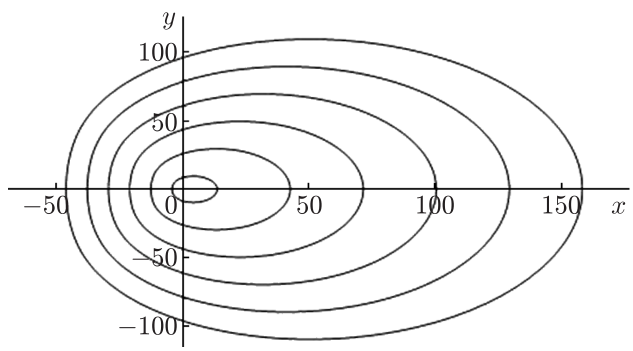

Fig. 3. Constant phase lines for waves of the first type (annular waves).

CONSTRUCTION OF ASYMPTOTICS FOR ANNULAR WAVES

We first consider annular waves—waves of the first type corresponding to the closed dispersion curve. The behavior of waves of this type is determined by the integrals J_2 at $x > 0$ and J_3 at $x < 0$. The integral J_2 is investigated (the integral J_3 is investigated similarly). We denote the phase by $\Phi_2 = \mu_2(\nu)x + \nu y - \omega t$. Next, we use the stationarity condition for the phase $\mu_2'(\nu) = -y/x$. At $x > 0$, these relations hold for the family of lines of the constant phase Φ_2 with the parameter ν:

$$x = \frac{\Phi_2 + \omega\, t}{\mu_2(\nu) - \nu\mu_2'(\nu)}, \qquad y = -\frac{\mu_2'(\nu)(\Phi_2 + \omega\, t)}{\mu_2(\nu) - \nu\mu_2'(\nu)}.$$

At $x < 0$, the family of constant phase lines is described by the same equations with the phase $\Phi_3 = \mu_3(\nu)x + \nu y - \omega t$.

Figure 3 shows the constant phase lines forming closed curves for a fixed value of t and for $\Phi_2 = \Phi_3 = 2\pi k$ ($k = 0.1, \ldots, 5$). We denote by ν_* the abscissa of the rightmost point of the closed curve in Fig. 1, at which $\mu_2(\nu_*) = \mu_3(\nu_*)$ (point with the abscissa $-\nu_*$ is the leftmost point). We represent the integral J_2 as the sum of two terms

$$J_2 = I_0 + I_1, \tag{7}$$

where

$$I_0 = -\frac{1}{2\pi}\int\limits_{-\nu_*}^{\nu_*} f_2(\nu)\,\mathrm{e}^{-i(\mu_2(\nu)x + \nu y - \omega t)}\, d\nu, \quad I_1 = -\frac{1}{2\pi}\int\limits_{-\nu_*}^{\infty} f_2(\nu)\,\mathrm{e}^{-i(\mu_2(\nu)x - \omega t)}\cos(\nu y)\, d\nu.$$

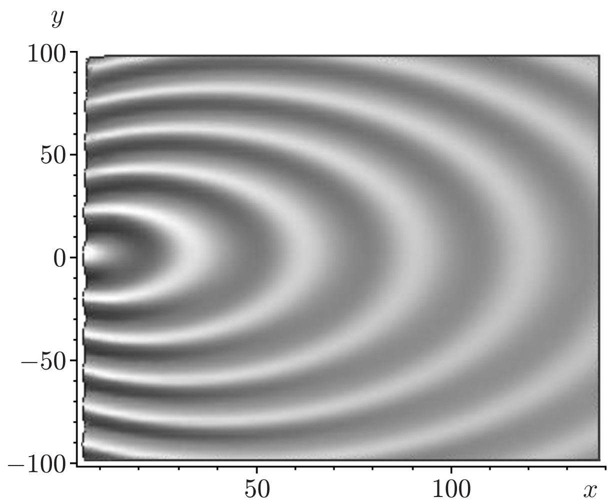

Fig. 4. Annular waves at $x > 0$.

For the function $\mu_2(\nu)$ and the denominator in the expression for function $f_2(\nu)$, the point ν_* is a branch point of the second order. The function $\mu_2(\nu)$ is analytically continued into the domain $|\nu| > \nu_*$ in such a way that the imaginary part $\mu_2(\nu)$ is negative. Consequently, in the neighborhood of the point ν_*, the functions $\mu_2(\nu)$ and $f_2(\nu)$ can be represented as $\mu_2(\nu) = \alpha + \beta\sqrt{\nu_* - \nu} + O(\nu_* - \nu)$ and $f_2(\nu) = (1/\gamma)\sqrt{\nu_* - \nu} + O(1)$. Then, the asymptotics of the integral I_1 for large values of x is calculated by integrating by parts:

$$I_1 = -\frac{2\sin(\alpha x - \omega t)\cos(\nu_* y)}{\pi\beta\gamma x} + O(x^{-2}). \tag{8}$$

The asymptotic behavior of the integral I_0 is calculated by the stationary phase method:

$$I_0 = -\frac{f_2(\nu(\rho))\cos(\mu_2(\nu(\rho))x + \nu(\rho)y - \omega t - \pi/4)}{\sqrt{-2\pi\mu_2''(\nu(\rho))x}} + O(x^{-1}). \tag{9}$$

Here $\nu(\rho)$ is the root of the equation $\mu_2'(\nu) = -\rho$; $\rho = y/x$; the remainder term $O(x^{-1})$ is determined by the contribution to the integral from the boundaries of the domain of integration and is equal to the leading term in decomposition (8) with the opposite sign. Therefore, the leading term of the asymptotics J_2 is the same as in (9), but the remainder term has the form $O(x^{-3/2})$. The function $\mu_2'(\nu)$ is mutually single-valued with the range $(-\infty, \infty)$, which significantly simplifies the determination of the inverse function $\nu(\rho)$ for real values of ρ by mathematical simulation using Mathematica type systems. The asymptotics of the integral J_3 is calculated similarly using the stationary phase method.

Asymptotics (9) also hold for small x (even for $x = 0$). Indeed, for small x and a fixed value of y (for example, $y > 0$) the stationary point ν_0 is located near the point ν_*: $\nu_0 = \nu_* - \beta^2/(4\rho^2)$. Then, asymptotics (9) becomes

$$J_2 = -\frac{\cos(\alpha x + \nu_* y - \omega t - \pi/4)}{\gamma\sqrt{\pi y}}. \tag{10}$$

Thus, asymptotics (10) is practically uniform in the region $x \geqslant 0$ for large values of $x^2 + y^2$.

Figure 4 shows the three-dimensional pattern of annular waves described by the integral J_2 in the region $x \geqslant 5$ calculated by the stationary phase method (9). We note that for large values of y, expression (10) is the real part of the asymptotics of the function

$$F(x, y) = -\frac{\sqrt{\nu_*}\,\mathrm{e}^{-i\alpha x - i\omega t}\,H_0^{(2)}(\nu_* y)}{\sqrt{2}\,\gamma}, \tag{11}$$

where $H_0^{(2)}$ is the Hankel function of the second kind.

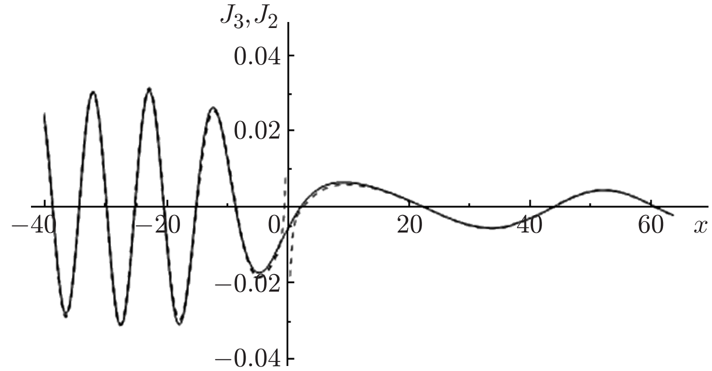

Fig. 5. Values of the integrals J_3 and J_2: the solid curve corresponds to the numerical calculation, and the dashed curve to the asymptotics.

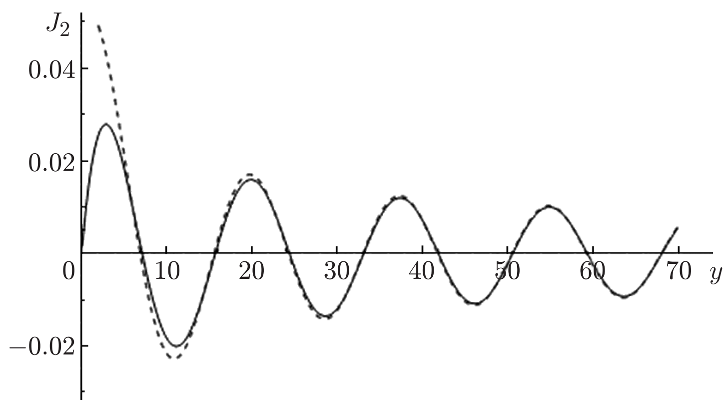

Fig. 6. The value of J_2 (solid curve) and the approximate value of J_2 determined by the Hankel function (dashed curve).

Figure 5 shows the values of the integrals J_2 and J_3 ($x < 0$) for $y = 25$ and $t = 10$ calculated by formula (7) (solid curve) and using the stationary phase approximation (dashed curve). It is seen that in front of the source of perturbations, the waves have a larger amplitude and a smaller length than behind the source. Figure 6 shows the integral J_2 for $x = 1$ and $t = 10$ calculated by formula (7) (solid curve) and formula (11) (dashed curve).

CONSTRUCTION OF ASYMPTOTICS FOR WEDGE-SHAPED WAVES

Waves of the second type (wedge-shaped waves) correspond to non-closed dispersion curves of $\mu_1(\nu)$ and $\mu_4(\nu)$. The behavior of these waves is completely determined by the integrals J_1 and J_4, respectively. Next, we consider the integral J_4. The boundary of the wave wedge is defined by the equality $y = \pm\mu_4(\nu_1)x$, where ν_1 is a root of the equation $\mu_4''(\nu) = 0$. Uniform asymptotics for integrals of this type were constructed in [13]. The uniform asymptotics for the integral J_4 at large distances from the moving oscillating source of perturbations has the form

$$J_4 = \frac{T_+(\rho)}{x^{1/3}}\ \mathrm{Ai}\,(x^{2/3}\sigma(\rho))\cos\,(a(\rho)x - \omega t) + \frac{T_-(\rho)}{x^{2/3}\sqrt{\sigma(\rho)}}\ \mathrm{Ai}'\,(x^{2/3}\sigma(\rho))\sin\,(a(\rho)x - \omega t), \qquad (12)$$

$$T_\pm(\rho) = \frac{1}{2}\left(f_4(\nu_2(\rho))\sqrt{\frac{-2\sqrt{\sigma(\rho)}}{S''_{\nu\nu}(\nu_2(\rho),\rho)}} \pm f_4(\nu_1(\rho))\sqrt{\frac{2\sqrt{\sigma(\rho)}}{S''_{\nu\nu}(\nu_1(\rho),\rho)}}\ \right)$$

$$\sigma(\rho) = \{(3/4)[S(\nu_2(\rho), r) - S(\nu_1(\rho), r)]\}^{2/3}, \qquad a(\rho) = [S(\nu_1(\rho), \rho) + S(\nu_2(\rho), \rho)]/2,$$

$$S(\nu, \rho) = \mu_4(\nu) + \rho\nu, \qquad \rho = y/x,$$

where $\nu_1(\rho)$, $\nu_2(\rho)$ are roots of the equation $S'_\nu(\nu, \rho) = 0$, $|\nu_2(\rho)| < |\nu_1(\rho)|$, $\mathrm{Ai}\,(\tau) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} \cos\left(\tau t - \frac{t}{3}\right)^3 dt$ is the Airy function, and $\mathrm{Ai}'(\tau)$ is the derivative of the Airy function. The nonuniform asymptotics describing wedge-shaped waves in the stationary phase approximation can be obtained from the uniform asymptotics if the Airy function and its derivative in (12) are replaced by their asymptotics for large positive values of the argument [3–5].

CONCLUSIONS

The asymptotic solutions constructed in this paper describe the amplitude–phase characteristics of the far fields of internal gravity waves generated by a local oscillating source of perturbations moving in a stratified flow of finite depth. It is shown that if the velocity of the perturbation source does not exceed the maximum group velocity of an individual wave mode, then the generated fields consist of waves of two types: annular and wedge-shaped. The obtained far-field asymptotics of annular and wedge-shaped waves can be used not only to calculate their main characteristics, but also to conduct a qualitative analysis of the solutions obtained. Wave patterns of these fields can be observed in remote sensing and measurements of internal gravity waves, generated by various sources of perturbations in natural and artificial stratified media.

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