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Hybrid Surface Waves from a Harmonic Perturbation Source

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Abstract—The problem of constructing uniform asymptotics of surface perturbations of far fields from a localized harmonic source in the flow of a heavy homogeneous fluid of infinite depth is considered. It is shown that the wave pattern of generated far fields at specific parameters is a system of hybrid wave disturbances that simultaneously possesses the properties of waves of two types: annular (transverse) and wedge-shaped (longitudinal) waves. The properties of the phase structure and wave fronts of the generated fields are studied. Uniform asymptotics of the solutions describing hybrid surface wave disturbances far from a harmonic source are constructed.

Keywords: heavy fluid, nonstationary source, surface disturbance, far fields, uniform asymptotic, transverse and longitudinal waves

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Surface wave motions in the marine environment can appear as a result of natural causes (wind waves; flow around undersea obstacles; and changes in the bottom topography, density fields, and currents) or they can be generated by a flow around artificial obstacles (platforms, underwater pipelines, and complex hydraulic constructions) [1–10]. The system of hydrodynamic equations that describes surface perturbations in the general form is a complex mathematical problem both in terms of proving the existence and uniqueness theorems for solutions in the corresponding functional classes; it is also a problem from the computational point of view [6, 7, 11]. We use methods of the Maslov integral representation; approximate methods of geometric optics; and, in some cases, the equations in conformal variables for the analytical investigation of surface wave disturbances within the framework of the linear theory as well [8, 12–14]. The main results of the solutions of the generation problems of surface wave disturbances are presented in the most general integral form. In this case, the integral solutions require the development of asymptotic methods for their analysis that permit qualitative analysis and make it possible to obtain estimates of these solutions [9, 10]. In addition, the analysis of remote sensing of a sea surface requires knowledge of the causes of specific surface phenomena [2, 8]. A detailed description of a wide range of physical phenomena associated with the dynamics of surface disturbances of inhomogeneous and nonstationary natural environment needs a basis of sufficiently developed mathematical models [6–8]. The three-dimensional structure of surface sea waves also plays an important role; at present, it is not possible to carry out large-scale computational experiments on modeling three-dimensional oceanic currents over long times with sufficient accuracy [6]. However, in a number of cases the initial qualitative representation of the investigated scope of wave phenomena can be obtained on the basis of simpler asymptotic models and analytical methods for their investigation. In this relation, it is necessary to note the classical problems of hydrodynamics on the construction of asymptotic solutions describing the evolution of surface perturbations excited by sources of various natures in a heavy homogeneous fluid [1, 3–5]. In particular, the boundaries of the wave zone and the field of sea surface elevation were calculated using the stationary phase method for some parameters of the generation of surface waves by a harmonic local source of disturbances [1]. Asymptotic solutions describing surface wave perturbations far from nonlocalized oscillating sources (bottom vibrations) were investigated in [14]. Model solutions make it possible to present surface wave fields in future, taking into account the variability and nonstationarity of the actual natural environment [9, 10]. A number of results from an asymptotic analysis of linear problems describing different modes of excitation and propagation of surface disturbances are the basis of the intensely developing nonlinear theory of the generation of ocean waves of extremely large amplitudes: rogue waves [6]. A review of the current state of investigations of linear and nonlinear surface wave perturbations is given in [8].

The problems of constructing uniform asymptotics of distant fields of internal and surface perturbations from a moving stationary source were considered in [15, 16]. Therefore, it is interesting to consider more complex regimes of the generation of surface waves caused by the nonstationarity of the source of perturbations.

The goal of this paper is to construct uniform asymptotics of long-range surface perturbations generated in the flow around a localized harmonic disturbance source by the flow of a heavy homogeneous fluid of infinite depth.

We consider the problem of a homogeneous flow of an infinitely deep heavy fluid around a harmonic source of perturbations with intensity $q = Q\exp(i\omega t)$. The velocity of the flow far from the source is V. The source is located at depth h relative to the unperturbed position of the free surface, i.e., at point $(0,0,-h)$. In order to find a physically realized solution of the problem, it is necessary to replace frequency ω by $\omega - i\varepsilon$, and then tend ε to zero in the found solution for the free surface. Further, let $\Omega(x,y,z,t)$ be the velocity perturbation potential with respect to the homogeneous flow. In the framework of the linear theory, we get the following problem to find Ω [3–5, 10].

$$\Delta\Omega(x,y,z,t) = Q\exp(i(\omega - i\varepsilon)t)\delta(x)\delta(y)\delta(z+h),\ z<0, \quad (1)$$
$$\left(\frac{\partial}{\partial t} + V\frac{\partial}{\partial x}\right)^2\Omega + g\frac{\partial\Omega}{\partial z} = 0,\ z = 0,$$

here, Δ is the three-dimensional Laplacian and $\delta(x)$ is the Dirac delta function. The free surface elevation $H(x,y,t)$ is determined from the Cauchy–Lagrange integral [3–5]

$$H(x,y,t) = -\frac{1}{g}\left(\frac{\partial}{\partial t} + V\frac{\partial}{\partial x}\right)\Omega(x,y,z,t),\ z = 0. \quad (2)$$

The solution of problem (1)–(2) is sought in the following form:

$$\Omega(x,y,z,t) = \exp(i(\omega - i\varepsilon)t)\varphi(x,y,z),$$
$$H(x,y,t) = \exp(i(\omega - i\varepsilon)t)\eta(x,y).$$

We get the following problem in dimensionless variables $x_* = gxV^{-2}$, $y_* = gyV^{-2}$, $z_* = gzV^{-2}$, $h_* = ghV^{-2}$, $\omega_* = V\omega/g$, $t_* = gt/V$, $\varepsilon_* = V\varepsilon/g$, $\varphi_* = V^2\varphi/Qg$, $\eta_* = V^3\eta/Qg$ to find functions $\varphi_*(x_*,y_*,z_*)$, $\eta_*(x_*,y_*)$ (hereinafter we omit the subscript *)

$$\Delta\varphi(x,y,z) = \delta(x)\delta(y)\delta(z+h),\ z<0,$$
$$\left(i\omega + \varepsilon + \frac{\partial}{\partial x}\right)^2\varphi + \frac{\partial\varphi}{\partial z} = 0,\ z = 0, \quad (3)$$

$$\eta(x,y) = -\left(i\omega + \varepsilon + \frac{\partial}{\partial x}\right)\varphi,\ z = 0. \quad (4)$$

We substitute functions $\varphi(x,y,z)$, $\eta(x,y)$ in the form of double Fourier integrals

$$\varphi(x,y,z) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\exp(-i\mu x)d\mu \times \int_{-\infty}^{\infty}\exp(-i\nu y)f(\mu,\nu,z)d\nu, \quad (5)$$

$$\eta(x,y) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\exp(-i\mu x)d\mu \times \int_{-\infty}^{\infty}\exp(-i\nu y)\Lambda(\mu,\nu)d\nu. \quad (6)$$

We substitute (5) into (3) and get the following boundary problem:

$$\frac{\partial^2 f(\mu,\nu,z)}{\partial z^2} - (\mu^2+\nu^2)f(\mu,\nu,z) = \delta(z+h),\ z<0,$$
$$(i\omega + \varepsilon - i\mu)^2 f(\mu,\nu,z) + \frac{\partial f(\mu,\nu,z)}{\partial z} = 0,\ z = 0. \quad (7)$$

Solution of problem (7) in domain $-h < z < 0$ is written as

$$f(\mu,\nu,z) = -\frac{(\omega\text{-}\mu)^2\sinh(kz) + k\cosh(kz)}{k\exp(kh)((\varepsilon + i(\omega-\mu))^2 + k)},$$
$$k^2 = \mu^2 + \nu^2.$$

Function $\Lambda(\mu,\nu)$ is found from (4)

$$\Lambda(\mu,\nu) = \frac{i(\omega-\mu)\exp(-kh)}{(\varepsilon + i(\omega-\mu))^2 + k}. \quad (8)$$

The denominator zeroes in (8) determine dispersion relation $(\omega-\mu)^2 = \sqrt{\mu^2+\nu^2}$, which can be written in the explicit form:

$$\nu(\mu) = \pm\sqrt{(\omega-\mu)^4 - \mu^2}. \quad (9)$$

Multitude of frequencies $\omega > 0$ is divided into three intervals by two characteristic values $\omega_1 = 0.25$ and $\omega_2 = \sqrt{6/9}$. At $\omega < \omega_1$, dispersion curve (9) consists of three branches: one closed branch and two unclosed ones. Then the wave pattern becomes a sum of two ship-bow (longitudinal) waves with a half-bow wave angle smaller than $\pi/2$ and annular (transverse) waves around the source. If $\omega > \omega_2$, dispersion curve (9) consists of two open branches without extrema. In this case, the wave pattern is a sum of two ship-bow waves with half-bow wave angle smaller than $\pi/2$. If $\omega_1 < \omega < \omega_2$, dispersion curve (9) consists of two open branches, one of which has two local extrema. One branch of the dispersion curve corresponds to the usual ship-bow waves with a half bow angle of the wedge smaller than $\pi/2$, while the second wave corresponds to ship-bow waves with a bow angle greater than $\pi/2$ (the wave front is directed upstream from the source). This system of hybrid waves combines the properties of the annular (transverse) and ship-bow (longitudinal) waves. Below, we shall consider this case precisely ($\omega = 0.255$).

Figure 1 shows the branch of dispersion relation (9) (denoted below as $\nu_1(\mu)$) describing hybrid waves. Figure 2 shows the branch of dispersion curve (9) (denoted below as $\nu_2(\mu)$) describing ship-bow waves. At $\varepsilon > 0$ and $\mu < \omega$, we get $\operatorname{Im}\nu_1(\mu) < 0$ and, in the case of $\mu > \omega$, $\operatorname{Im}\nu_2(\mu) > 0$. Then we calculate internal integral in (6) by closing the integration contour with respect to variable ν at $y > 0$ in the low half-plane (poles $\nu_1(\mu)$ and $-\nu_2(\mu)$), and at $y < 0$ in the upper half plane (poles $-\nu_1(\mu)$ and $\nu_2(\mu)$), and get

$$\eta(x, y) = I_1(x, y) + I_2(x, y).$$

$$I_1(x, y) = \frac{1}{2\pi}\int_{-\infty}^{C} \frac{E}{\nu_1(\mu)} \exp(-i(\mu x + \nu_1(\mu)|y|))d\mu,$$

$$I_2(x, y) = \frac{1}{2\pi}\int_{D}^{\infty} \frac{E}{\nu_2(\mu)} \exp(-i(\mu x - \nu_2(\mu)|y|))d\mu,$$

$$E = (\omega - \mu)^3 \exp(-(\omega - \mu)^2 h),$$

where C and D are the abscissas of the extreme right and left points of the dispersion curves $\nu_1(\mu)$ and $\nu_2(\mu)$ in Figs. 1 and 2, respectively. Points A and G are the inflection points of dispersion curves $\nu_1(\mu)$ and $\nu_2(\mu)$ in Figs. 1 and 2, respectively.

Integrals $I_2(x, y)$ with the corresponding dispersion relation $\nu_2(\mu)$, which describe usual ship-bow waves, are considered in detail, for example, in [9, 10, 15, 16]. A more complex and previously unstudied wave pattern of amplitude-phase characteristics of hybrid surface wave perturbations is described by integrals $I_1(x, y)$. Let us introduce notation for the phase: $\Phi = \nu_1(\mu)|y| + \mu x$. Then, we use the phase stationarity condition in the form of

$$\frac{d\nu_1(\mu)}{d\nu} = -\frac{x}{|y|}, \tag{10}$$

and get a family of constant phase lines with parameter μ (here we omit lower index 1)

$$x = \frac{\Phi\nu'(\mu)}{\nu(\mu) - \mu\nu'(\mu)}, \quad |y| = \frac{\Phi}{\nu(\mu) - \mu\nu'(\mu)}. \tag{11}$$

Figure 3 shows the constant phase line for different values of Φ with a step of 2π. Parts of the dispersion curve from point C to inflection point A correspond to annular (transverse) waves and a set of cusp points form a wave front upstream shown in Fig. 3 with a dashed line. Finding the wave front requires the need to add condition $\frac{\partial x}{\partial \mu}\frac{\partial |y|}{\partial \Phi} - \frac{\partial x}{\partial \Phi}\frac{\partial |y|}{\partial \mu} = 0$ to Eqs. (11), or $\frac{d^2\nu(\mu)}{d\mu^2} = 0$, which is the same, where abscissa μ_A of point A is a solution of this equation. Then the equation for determining the front would have the following form: $x = -\nu'(\mu_A)|y|$, and the corresponding half-angle of bow wave wedge would be equal to 105°7′. The parts of the dispersion curve from point A to point B correspond to the longitudinal wave crests from the front to infinity (they are shown in Fig. 3 to the left of the dashed line). The dashed line in Fig. 3 corresponds to the wave crest with phase $\Phi = 0$. It is described by equation $x = -\nu'(\mu_B)|y|$ or $x = \sqrt{16\omega^2 - 1}|y|$, where μ_B is a root of equation $\nu(\mu) = \mu\nu'(\mu)$, whose solution is $\mu_B = -\omega\nu'(\mu_B) = -\sqrt{16\omega^2 - 1}$. The parts of the dispersion curve to the left of point B correspond to the lon-

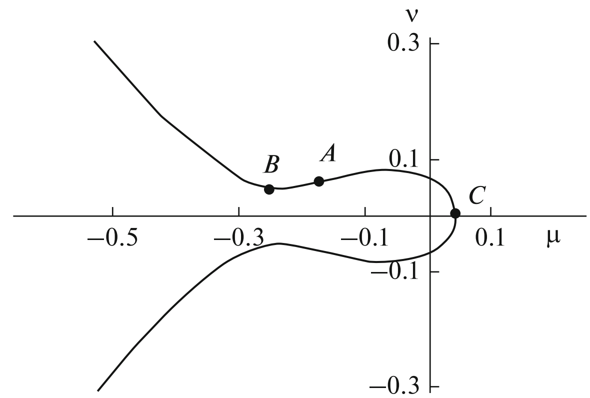

Fig. 1. Dispersion curve $\nu_1(\mu)$.

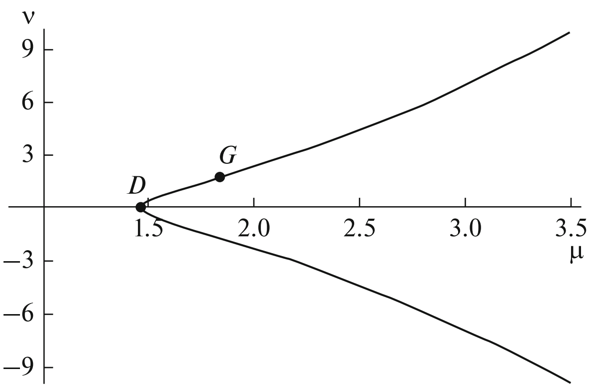

Fig. 2. Dispersion curve $\nu_2(\mu)$.

gitudinal wave crests propagating from infinity to the origin of the coordinates (to the right of the dashed line in Fig. 3). In this figure, phases Φ correspond to the region of the dispersion curve to the right of point B; their values are $2\pi n$, $n = 1, 2, 3, 4$, while the values of those to the left of point B are $2\pi n$, $n = -11, -10, .., -1$. In the infinity (at large values of x, $|y|$), the equations for the crests of longitudinal waves are written as $x = \sqrt{16\omega^2 - 1}|y| - 2\pi k/\omega$, where k is an integer number; i.e., these crests are the crests of a plane wave with length $\lambda = \pi\omega^{-2} \approx 24$. The wavelength of annular (transverse) waves in the direction of the x axis is $\lambda = 2\pi/C \approx 142$, which is approximately six times greater than the wavelength of the wedge-shaped (longitudinal) bow waves.

Integral $I_1(x, y)$ belongs to the class of integrals with two stationary points. The stationary points determined by Eq. (10) merge at the wave front (the dashed line in Fig. 3). Uniform asymptotics $I_1(x, y)$ at large values of $|y|$ is constructed similar to [15, 16]; it is written as

$$I_1(x,y) = \frac{T^+(\rho)}{|y|^{1/3}} Ai(|y|^{2/3}\sigma(\rho))\exp(-i|y|a(\rho))$$
$$- i\frac{T^-(\rho)}{|y|^{2/3}\sqrt{\sigma(\rho)}} Ai'(|y|^{2/3}\sigma(\rho))\exp(-i|y|a(\rho)),$$
$$T^{\pm}(\rho) = \frac{1}{2}\left(F(\mu_2)\sqrt{\frac{-2\sqrt{\sigma(\rho)}}{\theta(\mu_2,\rho)}} \pm F(\mu_1)\sqrt{\frac{2\sqrt{\sigma(\rho)}}{\theta(\mu_1,\rho)}}\right),$$
$$F(\mu) = (\omega-\mu)^3\exp(-h(\omega-\mu)^2)/\nu(\mu),$$
$$\sigma(\rho) = \left(\frac{3}{4}(S(\mu_2,\rho) - S(\mu_1,\rho))\right)^{2/3}, \qquad (12)$$
$$a(\rho) = \frac{1}{2}(S(\mu_2,\rho) + S(\mu_1,\rho)),$$
$$S(\nu,\rho) = \nu(\mu) - \rho\mu, \quad \rho = -x/|y|,$$
$$\theta(\nu,\rho) = \frac{\partial^2 S(\mu,\rho)}{\partial\mu^2},$$
$$Ai(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\cos\left(\tau u - u^3/3\right)du,$$

where $\mu_1 = \mu_1(\rho)$ and $\mu_2 = \mu_2(\rho)$ are the roots of equation $\partial S(\mu,\rho)/\partial\mu = 0$; in this case, $|\mu_1| < |\mu_2|$, $Ai(\tau)$ is the Airy function and $Ai'(\tau)$ is the derivative of the Airy function.

In order to get an expression for the free surface elevation, it is necessary to multiply expression (12) by $\exp(i\omega t)$ and take the real part of the result. Figure 4 shows the wave pattern of the free surface elevation at $t = 10$ and $h = 3$ calculated from (12) in dimensionless coordinates ($I_1(x, y)$ was calculated numerically in the neighborhood of the coordinate origin integral). At the values of amplitude parameters characteristic of the oceanic conditions $Q = 10^3$ m^3/s, $V = 3$ m/s, the surface elevation is on the order of 0.3 m. Using the asymptotics of the Airy function and its derivative far from the front, one can obtain nonuniform asymptotics for $I_1(x, y)$ consisting of two terms. The first of these two terms, corresponding to root μ_1, describes wedge-shaped (longitudinal) ship-bow waves, while the second, corresponding to root μ_2, describes annular (transverse) waves.

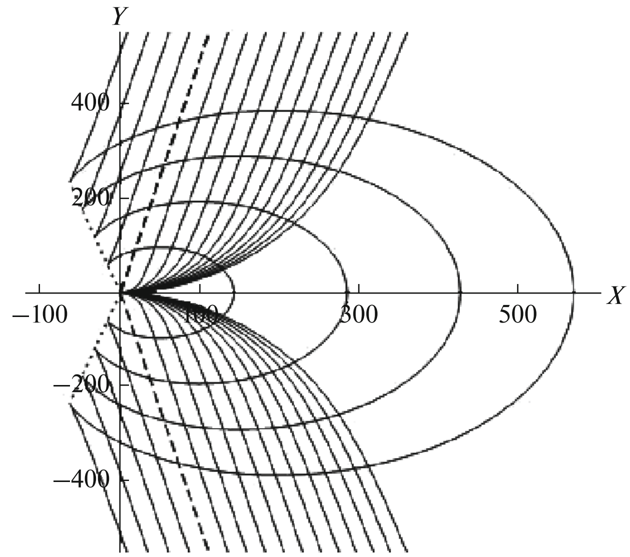

Fig. 3. Line of equal phase.

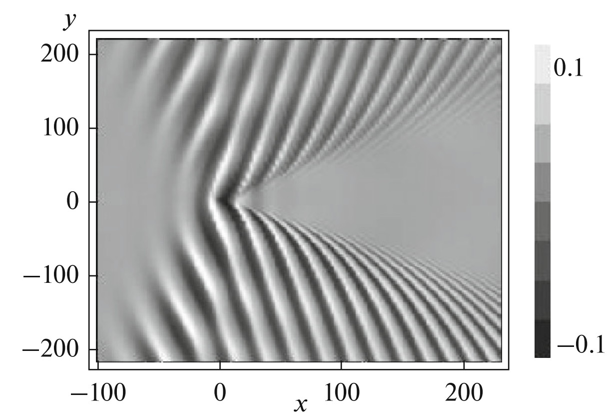

Fig. 4. Field of free surface elevation.

Thus, it is shown that the far fields of surface disturbances from a harmonic localized source in a heavy fluid flow of infinite depth under specific generation regimes have a form of hybrid waves of two types: annular (transverse) and wedge-shaped (longitudinal) waves. The nonstationarity of the amplitude of the disturbance source leads to the appearance not only of the ring waves diverging at the surface of the fluid directly from the source, but also to the generation of hybrid surface disturbances propagating upstream from the source. The asymptotics solutions of the long-wave fields of surface wave disturbances provide an efficient calculation of the main amplitude-phase characteristics of the generated wave fields and, in addition, make a qualitative analysis of the solutions possible, which is important for the correct formulation of the mathematical models of the wave dynamics of surface disturbances in the natural environment.

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