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Far Fields of Internal Gravity Waves from Oscillating Sources of Disturbances

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Abstract—We considered the problem dealing with the far field of internal gravity waves generated by the source oscillating in the vertical. It is shown that, at a fixed observation point, we first record the arrival of the main wave front of a separate mode followed by the arrival of two fronts of an additional field of this wave mode, which is caused by the oscillations of the source, the asymptotics near the additional wave front being expressed via Airy function.

Keywords: internal gravity waves, stratified medium, oscillating source.

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A detailed description of many of the physical phenomena associated with dynamics in natural stratified horizontally inhomogeneous and non-steady-state media necessitates examining quite advanced mathematical models which are, in general, very complex, nonlinear, and multiparametrical and can be adequately and efficiently explored only by means of numerical methods. However, initial information on the studied set of phenomena may sometimes be gained from simpler asymptotic models and analytical methods of their study, and the problems of dynamics of the internal gravity waves are very demonstrative in this regard [1–4].

To study all the wave effects, it is usually sufficient to construct relatively simple models amenable to a theoretical study and usable as a basis for studying and classifying in detail the processes of wave interaction in stratified media. However, despite the visible simplicity of the model assumptions, a successful choice for the form of the solution may lead to nontrivial and physically interesting results. In this regard we should mention the problem dealing with evolution of nonharmonic wave trains in smoothly inhomogeneous (in horizontal) and non-steady-state stratified medium [1, 2].

The constructed model solutions, or anzatz, describing the structure of the fields near the wave fronts of separate modes in the vertically stratified medium then make it possible to describe the asymptotic representations of the fields of internal waves, taking into account the real variations in the ocean. Moreover, it was found that the constructed asymptotic solutions compare well with results of field observations of the internal waves in the ocean [5, 6].

Therefore, it is interesting to obtain solutions which describe the far fields of internal waves from oscillating sources of disturbances, which will be used subsequently as a basis for constructing the corresponding asymptotic anzatz. An explicit identification of effects determined by the oscillations of the source of disturbances in stratified flows will then make it possible to obtain solutions that describe the wave dynamics of the internal gravity waves in the real natural stratified media. A far-region wave field in the case of resonance in the two-dimensional approximation was considered in [7]; wave fields from arbitrarily moving and, in particular, oscillating sources of disturbances were studied numerically in [1, 2].

This work considers the source of disturbances, which moves uniformly along a certain axis in the layer of arbitrarily stratified medium $-H < z < 0$ and periodically oscillates with frequency ω_0 vertically. Below we will study the characteristics of the far field of the internal gravity waves in this case and their connection with the case of the uniform rectilinear motion of the source. The point source of disturbances moving horizontally at the speed V will be considered to switch on at $t = 0$; then the wave field G, excited by this oscillating source, satisfies the equation [1–4]

$$LG = \delta'(x + Vt)\delta(y)\delta(z - z_0 - A\cos\omega_0 t)\Theta(t),$$

$$L = \frac{\partial^2}{\partial t^2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) + N^2(z)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right),$$

$$G = 0, \quad z = 0, \ -H,$$

where $N^2(z)$ is the Brunt–Vaisala frequency, $\Theta(t) = 0$ for $t < 0$, $\Theta(t) = 1$ for $t > 0$, L is the operator of internal gravity waves in the Boissinesq approximation, z_0 is the average depth of the source of disturbances, and A

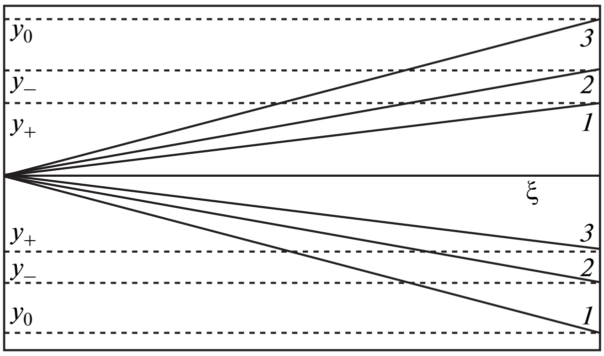

Fig. 1. Wave fronts of the first mode from moving the oscillating source of disturbances: (*1*) wave front of the main field U_1^0 and wave fronts of the additional fields (*2*) U_1^+ and (*3*) U_1^-.

is the oscillation amplitude of the source. The limit of the function G will be sought for fixed $\xi = x + Vt$ and at $t \to \infty$.. Let Γ be a function determined from the problem $L\Gamma(x, y, z, z_0, t) = \delta'(x)\delta(y)\delta(z - z_0)\delta(t)$, $\Gamma \equiv 0$ $(t < 0)$, $\Gamma = 0$, $z = 0, -H$. Then, $G = \int_0^t \Gamma(x + V\tau, y, z, z_0 + A\cos\omega_0\tau, t - \tau)\,d\tau$. The function Γ is decomposed into the sum of the modes $\Gamma = \sum \Gamma_n$, where

$$\Gamma_n = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\cos(\mu x + \lambda y)\sin(\omega_n(k)t)\,\mu\omega_n(k)\varphi_n(z,k)\varphi_n(z_0,k)}{8\pi^2 k^2} d\mu d\lambda.$$

Here, $k = \sqrt{\lambda^2 + \mu^2}$, $\omega_n(k)$ is the variance curve of the nth mode, and $\varphi_n(z, k)$ is the corresponding eigenfunction of the main vertical spectral problem of the internal waves [1–4]:

$$\frac{\partial^2\varphi_n(z,k)}{\partial z^2} + k^2\left(\frac{N^2(z)}{\omega_n^2(k)} - 1\right)\varphi_n(z,k) = 0,$$

$$\varphi_n = 0, \quad z = 0, -H.$$

The function G can also be represented as a sum of wave modes G_n, where

$$G_n = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^t \frac{\Phi_n \mu\omega_n(k)\varphi_n(z,k)\varphi_n(z_0 + A\cos\omega_0\tau, k)}{8\pi^2k^2} d\tau d\mu d\lambda,$$
$$\Phi_n = \cos(\mu(x + V\tau) + \lambda y)\,\sin(\omega_n(t - \tau)). \tag{1}$$

We expand the function $\varphi_n(z_0 + A\cos\omega_0\tau, k)$ in the Fourier series over τ: $\varphi_n(z_0 + A\cos\omega_0\tau, k) = \Psi_0(z_0, k) + A\Psi_1(z_0, k)\cos\omega_0\tau + \ldots$ [1–3]. Substituting the first term of this expansion in (1), we see that each wave mode consists of the main field G_n^0, arising in the case of the uniform motion of the source of disturbances, plus the field G_n^1, which is caused by the oscillations of this source. Generally, a few low-order wave modes are actually excited in the ocean; hence, $\varphi_n(z, k)$ can be considered a slowly varying function in variable z [1, 2, 4, 5]. Therefore, in (1) we let $\Psi_0(z_0, k) \approx \varphi_n(z_0, k)$, $\Psi_1(z_0, k) \approx \dfrac{\partial\varphi_n(z_0, k)}{\partial z_0}$. Then we obtain

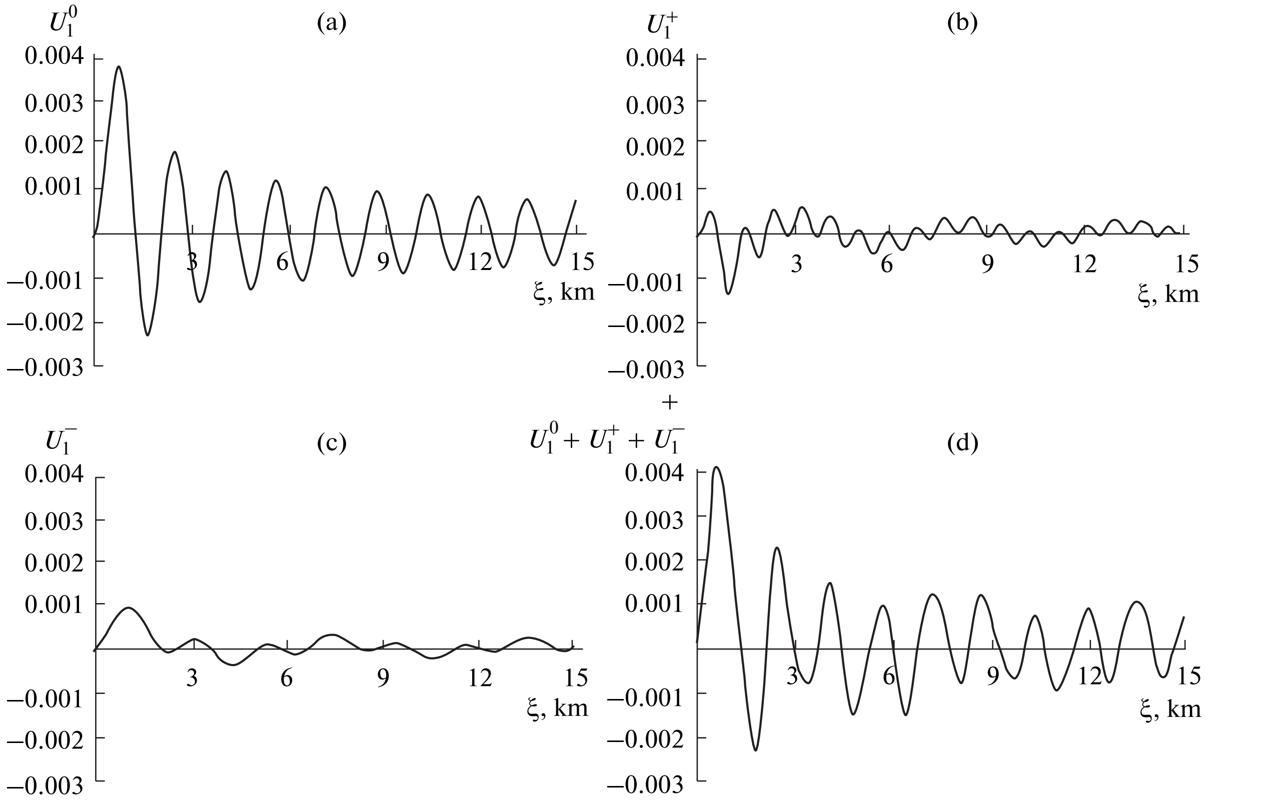

Fig. 2. First mode of the far field of the internal gravity waves from moving oscillating source of disturbances: (a) function U_1^0, (b) function U_1^+, (c) function U_1^-, and (d) the sum $U_1^0 + U_1^+ + U_1^-$.

$$G_n^1 = A\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{t}\frac{\Phi_n\mu\omega_n(k)\varphi_n(z,k)\cos\omega_0\tau}{8\pi^2k^2}\frac{\partial\varphi_n(z_0,k)}{\partial z_0}d\tau d\lambda d\mu.$$

Taking the integral over the variable τ, we have

$$G_n^1 = A\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{(\Omega_n^+ + \Omega_n^-)\cos(\mu x + \nu y)\omega_n(k)\varphi_n(z,k)\mu}{8\pi^2k^2}\frac{\partial\varphi_n(z_0,k)}{\partial z_0}d\lambda d\mu,$$

$$\Omega_n^{\pm} = \frac{\sin(\mu Vt \mp \omega_0 t) + \sin(\omega_n t)}{\mu V - \omega_n \pm \omega_0}.$$

We shift the integration contour over μ toward the upper half plane; then, for fixed $\xi = x + Vt$ and as $t \to \infty$, we can obtain

$$\lim_{\substack{t\to\infty\\ \xi = x+Vt = \mathrm{const}}} G_n^1$$

$$= A\int_{-\infty}^{\infty}\int_{-\infty+i\varepsilon}^{\infty+i\varepsilon}\frac{(Q_n^+ + Q_n^-)\mu\omega_n(k)\varphi_n(z,k)}{4\pi^2k^2}\frac{\partial\varphi_n(z_0,k)}{\partial z_0}d\lambda d\mu,$$

$$Q_n^{\pm} = \frac{\sin(\mu\xi + \nu y \mp \omega_0 t)}{\mu V - \omega_n \pm \omega_0}.$$

For $\xi < 0$, this integral is exponentially small; for $\xi > 0$, the integration should be translated to the lower half plane [1, 2]. As a result, we obtain

$$G_n^1 = G_n^+ + G_n^-, \qquad (2)$$

$$G_n^{\pm} = A\int_{-\infty}^{\infty}\frac{\cos(\mp i\omega_0 t - \mu_{n\pm}\xi - \lambda y)}{2\pi(V - c_n\mu_{n\pm}/k)}D_n^{\pm}(z,k)d\lambda,$$

$$D_n^{\pm}(z,k) = \frac{\mu_{n\pm}\omega_n(k)\varphi_n(z,k)}{k^2}\frac{\partial\varphi_n(z_0,k)}{\partial z_0},$$

where $\mu_{n\pm} = \mu_{n\pm}(\lambda)$ are the solutions of the corresponding equations: $\mu_{n\pm}V = \omega_n(k) \pm \omega_0$, $k = \sqrt{\lambda^2 + \mu_{n\pm}^2}$, $c_n = \partial\omega_n/\partial k$. The integrals (2) for large y, ξ can be calculated asymptotically, provided that the properties of the functions $\mu_{n\pm}(\lambda)$ are known [1, 2, 8]. The maximum of $\partial\mu_{n\pm}/\partial\lambda$ determines the positions of the wave fronts far from the oscillating source of disturbances; the field near these fronts can be expressed via the Airy function [8]. The additional field G_n^1, which arises due to the oscillations of the source of disturbances, should be compared to the main field G_n^0, of the form

$$G_n^0 = \int_{-\infty}^{\infty} \frac{\cos(\mu_{n0}\xi - \lambda y)}{2\pi(V - c_n\mu_{n0}/k)} B_n(z,k)d\lambda, \quad (3)$$

where $\mu_{n0} = \mu_{n0}(\lambda)$ is the solution of the equation $\mu_{n0}V = \omega_n(k)$, $k = \sqrt{\lambda^2 + \mu_{n0}^2}$, $D_n^{\pm}(z,k)/B_n(z,k) = A\frac{\partial\varphi_n(z_0,k)}{\partial z}\Big/\varphi_n(z_0,k)$. The function $\partial\mu_{n0,\pm}/\partial\lambda$ reaches a maximum at the point λ_0 and $\mu_{n0,\pm} = \mu_{n0,\pm}(\lambda_0)$, and the function $\mu_{n0,\pm}(\lambda)$ near this point λ_0 is expanded in series over odd powers of $\lambda - \lambda_0$: $\mu_{n0,\pm}(\lambda) = \mu_{n0,\pm} + q_{n0,\pm}(\lambda - \lambda_0) - -b_{n0,\pm}(\lambda - \lambda_0)^3 + \ldots$ [1,2]. Obviously the positions of wave fronts for separate modes are determined as $y = \xi q_{n0,\pm}$. Then, using expansions of the functions $\mu_{n0\pm}(\lambda)$ around the point λ_0, we can determine the asymptotics $U_n^0, U_n^{\pm}$ of the integrals (2) and (3) near the wave fronts far from the sources of disturbances of separate modes, which have the form [1, 2, 8]

$$U_n^{0,\pm} \approx \frac{\cos(M^{0,\pm}\omega_0 t + \mu_{n0,\pm}\xi)}{V - c_n(k_0)\dfrac{\mu_{n0,\pm}}{k_0}} \frac{R_n^{0,\pm}}{\sqrt[3]{3b_{n0,\pm}\xi}} \quad (4)$$
$$\times Ai\left(\frac{\xi q_{n0,\pm} - y}{\sqrt[3]{3b_{n0,\pm}\xi}}\right)\frac{\mu_{n0,\pm}\omega_n(k_0)}{k_0^2}\varphi_n(z,k_0)\frac{\partial\varphi_n(z_0,k_0)}{\partial z_0},$$

where $M^0 = 0$, $M^{\pm} = \pm 1$, $k_0 = \sqrt{\mu_{n0,\pm}^2 + \lambda_0^2}$, $Ai(x)$ is the Airy function [8], and $R_n^{0,\pm}$ are the corresponding amplitude factors. Numerical estimates were made using a typical nonconstant distribution $N(z)$, taken from [5]. Other parameters had the following values: $\omega_0 = 2\pi/T$, $T = 10$ min, $V = 4$ m/s, and $A = 1$ m. For these parameter values, the field asymptotics U_n^0 and $U_n^{\pm}$ for the first and second wave modes were compared.

The positions of the wave fronts are schematically illustrated in Fig. 1, and the calculations of the wave field of the first mode from formulas (4) for $y = 100$ m are presented in Fig. 2. Thus, the far field of the internal gravity waves from moving oscillating source of disturbances has the following qualitative pattern. The wave front of a separate mode of the main field comes first to a fixed point, and then two fronts of the additional field of this wave mode, caused by the oscillations of the source, come in successively. The amplitude of the wave field near additional wave fronts usually does not exceed 10% of the amplitude of the main wave field. The obtained wave pattern qualitatively coincides with the results of an exact numerical solution of this problem [1, 2]; therefore, the asymptotic representations of the solution can be used in the future for determining the far-region wave fields.

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