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Wave Field Dynamics of a Stratified Medium with Variable Depths

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The distribution of inner gravitational waves in the ocean is substantially influenced by both heterogeneities in hydrophysical fields (for example, density field) and variations in the bottom topography. Moreover, correct analytical solutions of wave problems result only in a situation, when the water density distribution and bottom topography are described by relatively simple model functions. When the medium and boundary parameters are unconditioned, only numerical solutions of such problems are possible. At the same time, the latter prevent the adequate analysis of characteristics of wave fields, particularly over a large distance, which is necessary for solving some problems, such as, for example, detection of inner waves by remote methods including aerospace radiolocation. In such a situation, the description and analysis of wave dynamics may be realized through developing asymptotic models and using analytical methods for their solution based on the proposed modified method of geometrical optics.

In this communication, we consider the problem of far fields of inner gravity waves that propagate in the sea medium with a finite depth and variable bottom topography. The heterogeneity of the density field is modeled by the constant distribution of the Brent-Vaisala frequency ($N = \text{const}$).

In the linear approximation, the system of equations describing small-amplitude variations in the medium initially at rest (axis z is directed vertically downward) acquires the following form [1]:

$$\operatorname{div} \mathbf{U} = 0,$$

$$\rho_0 \frac{\partial U}{\partial t} + \operatorname{grad} p + G = 0, \qquad (1)$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho_0}{\partial z} w = 0,$$

where $\mathbf{U} = (u_1, u_2, w)$, p, ρ are perturbations of the velocity, pressure, and density vectors, respectively; $\rho_0(z)$ is the density of the medium in the unperturbed state; $G = (0, 0, g\rho)$; and g is the gravitational acceleration. The Boussinesq approximation is used for considering a water layer with the parameter $N^2(z) = -\frac{g d \ln \rho_0}{dz} = \text{const}$ bounded by a "solid cover" ($w = 0$) from above ($z = 0$) and below $z = -H(y)$, which satisfies the nonpercolation condition $w + \frac{dH}{dy} u_2 = 0$. It should be noted that these approximations adequately describe the main parameters of the sea medium in shelf areas of the Arctic Ocean [2, 3].

In real sea conditions, it may be assumed that the function $H(y)$ is a smooth gradually changing function. The variation smoothness of $H(y)$ means that the ratio between the characteristic horizontal scale L of variations $H(y)$ and characteristic vertical scale M of variations in amplitudes of the inner waves is determined as $\lambda = \frac{L}{M} \gg 1$, which means, in fact, a gentle incline of the bottom. In these conditions, the bottom relief heterogeneity may be modeled by a single elevation, i.e., the function $H(y)$ can be represented with a single maximum.

Solution (1) for the vertical velocity is obtained in the form

$$w = \exp(-i\omega t + ilx) W(z, y),$$

where ω is the frequency, and l is the horizontal wave number along the axis x, i.e., orthogonally to the waveguide. As far as the problem is linear, more general solutions are obtained by superposition of the

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obtained asymptotic representations. With dimensionless variables

$$x^R = \frac{x}{L}, \quad y^R = \frac{y}{L}, \quad z^R = \frac{z}{M}, \quad l^R = lM,$$

$$\omega^R = \frac{\omega}{N}, \quad h(y) = \frac{H(Ly)}{M}$$

(further, index R is omitted), we obtain

$$\frac{\partial^2 W}{\partial z^2} - \frac{1}{\lambda^2 c^2}\frac{\partial^2 W}{\partial y^2} + \frac{l^2}{c^2}W = 0,$$

$$W = 0 \quad \text{under} \quad z = 0, \tag{2}$$

$$W + \frac{dh(y)}{\lambda dy}u_2 = 0 \quad \text{under} \quad z = -h(y), \quad c^2 = \frac{\omega^2}{1-\omega^2}.$$

When the bottom profile is linear $\left(z = -\gamma y,\ \gamma = \frac{1}{\lambda}\right.$ is a bottom incline$\left.\right)$, problem (2) in the zero approximation (i.e., the boundary condition at the bottom acquires the form $W = 0$ with $z = -h(y)$) has an analytical solution [5]:

$$W = \sum W_n,$$

$$W_n = e^{-i\cdot\pi\nu/2}K_\nu\left(l\sqrt{\lambda^2 y^2 - \frac{z^2}{c^2}}\right)\sin\left(\frac{n\pi}{\ln\Delta}\ln\frac{\lambda c y - z}{\lambda c y + z}\right),$$

where $\Delta = \frac{\lambda c + 1}{\lambda c - 1}$, $\nu = \frac{2\pi n i}{\ln\Delta}$, K_ν is the Macdonald function of the imaginary index ν.

When the bottom profile differs from the linear one, solution (2) is sought in the form typical of the method of geometrical optics [4]:

$$W = \left(F_0(z, y, \omega) + \frac{i}{\lambda}F_1(z, y, \omega) + \left(\frac{i}{\lambda}\right)^2 F_2(z, y, \omega) + \ldots\right)e^{i\lambda S(y,\omega)},$$

where functions F_0 and F_1 are derived from equations

$$\frac{\partial^2 F_0}{\partial z^2} + \left(\left(\frac{\partial S}{\partial y}\right)^2 + l^2\right)\frac{F_0}{c^2} = 0,$$

$$F_0 = 0 \quad \text{at} \quad z = 0,\ -h(y), \tag{3}$$

$$\frac{\partial^2 F_1}{\partial z^2} + \left(\left(\frac{\partial S}{\partial y}\right)^2 + l^2\right)\frac{F_1}{c^2} = \frac{1}{c^2}\left(2\frac{\partial F_0}{\partial y}\frac{\partial S}{\partial y} + F_0\frac{\partial^2 S}{\partial y^2}\right),$$

$$F_1 = 0 \quad \text{at} \quad z = 0, -h(y).$$

The solution of the first equation of (3) provides the mode structure of the solution with the dispersion relation:

$$\kappa_n^2(y, \omega) = \frac{c^2 n^2 \pi^2}{h^2(y)}, \quad n = 1, 2, \ldots,$$

and eigenfunctions in the zero approximation (vertical modes)

$$F_{0n}(z, y, \omega) = A_{0n}(y, \omega)\sin\frac{n\pi z}{h(y)}, \quad n = 1, 2, \ldots$$

In this case, an eikonal $S_n(y, \omega)$ is obtained using the following equation:

$$\kappa_n^2(y, \omega) = \left(\frac{\partial S_n}{\partial y}\right)^2 + l^2.$$

The amplitude $A_{0n}(y, \omega)$ is derived from the solvability condition of the second equation from (3), which requires orthogonality between the right part of this equation and the function F_{0n}:

$$A_{0n} = \frac{B_{0n}(y_0, \omega)}{\sqrt[4]{c^2 n^2 \pi^2 - h^2(y) l^2}},$$

where the constant B_{0n} depends on ω and the initial eikonal value at any point y_0, $S_n(y_0, \omega)$. The eikonal $S_n(y, \omega)$ is described by the formula

$$S_n(y, \omega) = \int_y^{y^*}\sqrt{\kappa_n^2(y, \omega) - l^2 dy},$$

where y^* is the turning point, i.e., the root of equation $\kappa_n^2(y, \omega) = l^2$.

The geometrical position of turning points determines the caustic surface, near which properties of wave fields experience qualitative changes reflected in the transition from the "light" area, i.e., the area of wave fields, to the "shadow" area, where these fields are characterized by exponentially low values. In this case, the solution in the approximation of geometrical optics for a particular wave mode before the turning point, i.e., in the wave zone, acquires the following form:

$$W_n = \sqrt{2\pi}Q_n\cos\left(\lambda S_n(y, \omega) - \frac{\pi}{4}\right),$$

$$Q_n = \frac{\sin\frac{n\pi z}{h(y)}}{\lambda^{1/2}\sqrt[4]{c^2 n^2 \pi^2 - h^2(y) l^2}}.$$

After the turning point (in the zone of exponential attenuation), this solution may be

$$W_n = \sqrt{\pi}Q_n\exp(-\lambda|S_n(y, \omega)|).$$

In this case, the uniform asymptotics applied near the turning point is described by the formula

$$W_n = \sqrt{2}\pi\left(\frac{3}{2}\lambda S_n(y, \omega)\right)^{1/6}Q_n\mathrm{Ai}\left(\left(\frac{3}{2}\lambda S_n(y, \omega)\right)^{2/3}\right),$$

where $\mathrm{Ai}(x)$ is the Airy function.

Figures 1 and 2 present the results of calculations of the vertical velocity for two typical profiles of the oceanic bottom that differ from their linear counterparts observed in the Arctic basin [2, 3]. Figure 1 presents the calculation results for the level line of the first mode $W_1(z, y)$ in the model of the gradually sloping bottom surface. In this case, $h(0) = 0$ and any ω value under the assigned wave number l is characterized by

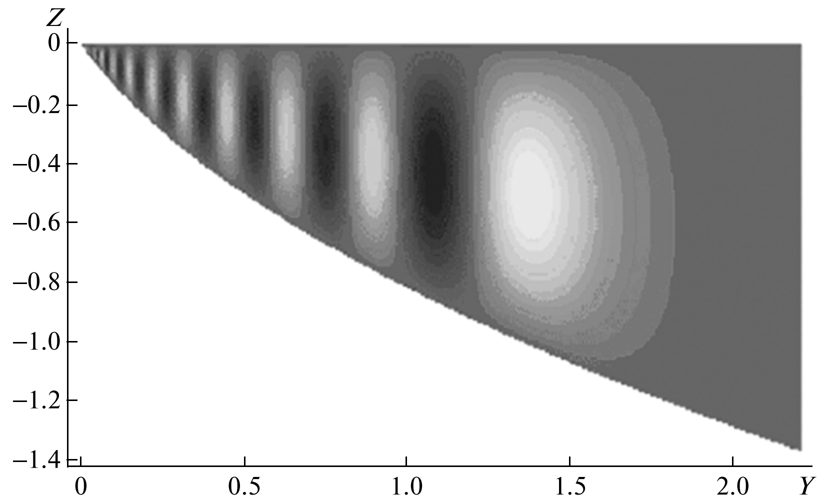

Fig. 1. The first mode of the vertical velocity above the sloping bottom profile.

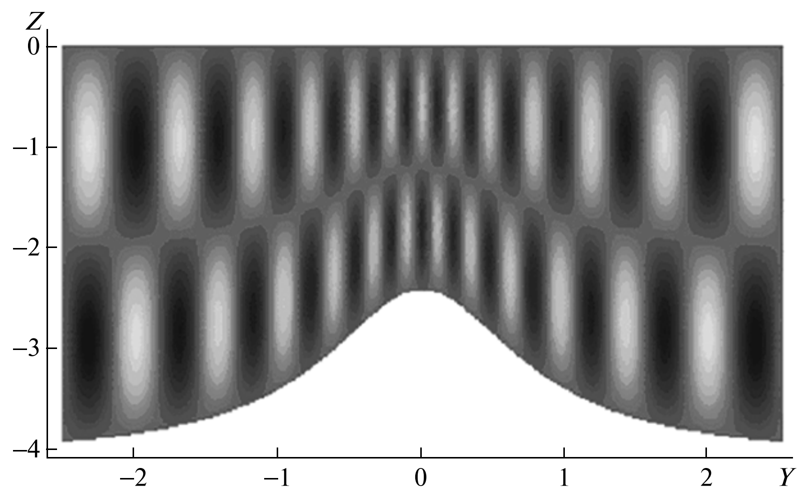

Fig. 2. The second mode of the vertical velocity above the solitary elevation.

the turning point y^* and the existence of only captured waves. Figure 2 presents the calculation results for the level line of the second mode $W_2(z, y)$ for the bottom profile with smooth elevation. In such a situation, $h(0) = h_0 \neq 0$, $h(\infty) = h_\infty$. This means that the cutoff frequency

$$\omega_0 = \Omega(h(0)),\ \ \Omega(h) = \frac{lh}{\sqrt{n^2\pi^2 + l^2h^2}},$$

is characterized by a value such that waves with frequencies $\omega < \omega_0$ do not exist. The frequency $\omega_0 < \omega < \omega_*$ ($\omega_* = \Omega(h_\infty)$) results in the appearance of a discrete spectrum, where each frequency ω_n corresponds to the captured wave. Under $\omega_* < \omega < 1$, turning points are missing, the ω spectrum is continuous, and progressive waves are developed.

Thus, it is shown that the bottom topography and structure of the stratified sea medium determine different parameters of far fields of inner waves. The effect of the space–frequency "blockage" of the wave field is characteristic of the real oceanic shelf. Depending on the frequency characteristics of the wave field and the bottom relief geometry, far fields of inner waves are either localized in some limited spatial domain (captured waves) or propagate in absence of turning points over relatively long distances as compared with the water depth (progressive waves). The spatial domain, where progressive waves propagate, is entirely determined by the existence of turning points, the location of which depends on the medium stratification and heterogeneities in the bottom topography.

The obtained asymptotic solutions are uniform and allow far fields of inner waves to be described both near and far from turning points. The universal character of the asymptotic method proposed for modeling far fields of inner waves makes it possible to effectively calculate wave fields and, in addition, qualitatively analyze the obtained solutions. This method offers broad opportunities for the analysis of wave fields on a large scale, which is important for developing correct mathematical models of wave dynamics and for assessing in-situ measurements of wave fields in the sea medium. The particular role of the proposed asymptotic methods is determined by the fact that the parameters of natural stratified media are usually known approximately and attempts at their adequate numerical solution using the initial equations of hydrodynamics and such parameters may result in a notable loss of accuracy for the results obtained. In addition to their fundamental significance, the obtained asymptotic models are also important for applied investigations, since the proposed method of geometrical optics allows solution of a wide spectrum of problems related to modeling wave fields.

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