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Estimate of the Applicability Limits of a Linear Theory of Internal Waves

V. V. Bulatov and Yu. V. Vladimirov

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Abstract—Using a perturbation method, the applicability limits of a linear theory of internal gravity waves are estimated. It is shown that over a wide range of wavelengths, typical of a real ocean, in studying the dynamics of internal gravity waves it is possible to use a linear approximation, which confirms the validity and adequacy of this approximation for the corresponding spatial and temporal scales of the linear model of wave dynamics.

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Internal gravity waves, being a specific case of wave motion of nonuniform media, are as a rule studied using universal mathematical methods, which provide analogies with waves of different physical nature [1, 2]. This approach makes it possible to obtain a formal answer but do not guarantee the adequacy of the solution of actual fundamental and applied problems of wave dynamics in natural stratified media (ocean, atmosphere) [3–5]. The specific ratios of spatial and temporal scales, for instance, in the ocean and the difficulties associated with this necessitate the investigation of the applicability limits of the linear theory of internal gravity waves, undertaken in this work. The excitation and propagation of internal gravity waves in real conditions are essentially nonlinear phenomena; however, under certain assumptions, the equations of generation and propagation of internal waves can be linearized [3, 5–8]. In [1, 2], a linear approximation was used for studying the wave dynamics of internal gravity waves in stratified media. It is of interest to estimate the adequacy of the assumptions used in those studies for real spatial and temporal geophysical scales.

1. BASIC EQUATIONS

In the adiabatic approximation for the equation of state, the system of hydrodynamic equations with account for the nonlinearity, viscosity, and Earth rotation take the form [3, 5, 9]:

$$\rho \frac{dU_1}{dt} + \frac{\partial p}{\partial x} = -\rho f U_2 + \nu \left[\Delta_3 U_1 + \frac{\partial}{\partial x} \operatorname{div} \mathbf{U} \right] + F_x,$$

$$\rho \frac{dU_2}{dt} + \frac{\partial p}{\partial x} = -\rho f U_1 + \nu \left[\Delta_3 U_2 + \frac{\partial}{\partial y} \operatorname{div} \mathbf{U} \right] + F_y,$$

$$\rho \frac{dW}{dt} + \frac{\partial p}{\partial x} + g\rho = \nu \left[\Delta_3 W + \frac{\partial}{\partial z} \operatorname{div} \mathbf{U} \right] + F_z,$$

$$\frac{1}{c^2} \frac{dp}{dt} = \frac{d\rho}{dt}, \tag{1.1}$$

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \rho \mathbf{U} = M,$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + U_1\frac{\partial}{\partial x} + U_2\frac{\partial}{\partial y} + W\frac{\partial}{\partial z}, \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad \Delta_3 = \Delta + \frac{\partial^2}{\partial z^2},$$

$$f = 2\Omega\sin\theta.$$

Here, $\Omega = 7.27\times 10^{-5}$ s^{-1} is the Earth angular velocity, θ is the geographic latitude, $\nu = 10^2$ cm^2/s is the viscosity coefficient, $\mathbf{U} = (U_1,\ U_2,\ W)$ are the velocity components, p and ρ are the pressure and density, the z axis is directed vertically upward, $g = 980$ cm/s^2 is the gravity force acceleration, $c = 1.5\times 10^5$ cm/s^2 is the sonic velocity in the ocean, F_x, F_y, and F_z are the densities of the body force exerted on the fluid, and M is the mass source density [3, 5, 9].

Below, we will consider either a medium unbounded in the vertical direction or a layer bounded by the bottom $z = -H$ and the free surface $z = \zeta(x,\ y,\ t)$. In the presence of viscosity, on the bottom we should specify the no-slip condition $U_1 = U_2 = W = 0$ at $z = -H(x,\ y)$, which will form the corresponding boundary layer. However, for internal and gravity waves in the ocean, typically the fluid particle velocities are small (of the order of 10 cm/s or even less), and hence the velocity gradients are also small. Accordingly, on the bottom we will use the no-flow condition [5–8]

$$W - U_1\frac{\partial H}{\partial x} - U_2\frac{\partial H}{\partial y} = 0 \quad (z = -H(x,\ y)). \tag{1.2}$$

For a horizontal bottom ($H = \text{const}$) this condition is simplified:

$$W = 0 \quad (z = -H). \tag{1.3}$$

On the ocean free surface $z = \zeta(x,\ y,\ t)$, two boundary conditions, kinematic and dynamic, are specified. The kinematic condition requires the normal to the surface velocity component $(U_1,\ U_2,\ W)$ to coincide with the surface displacement velocity:

$$z = \zeta:\ W = \frac{\partial \zeta}{\partial t} + U_1\frac{\partial \zeta}{\partial x} + U_2\frac{\partial \zeta}{\partial y} = \frac{d\zeta}{dt}. \tag{1.4}$$

The dynamic condition requires the pressure on the surface to coincide with the atmospheric pressure, which in what follows is set equal to zero [5–8]:

$$p(x,\ y,\ \zeta(x,\ y,\ t)) = 0. \tag{1.5}$$

We will linearize the system of Eqs. (1.1) and boundary conditions (1.2)–(1.5) relative to the state of rest:

$$U_1 = U_2 = W = 0, \qquad \rho = \rho_0, \qquad p = p_0(z) = -g\int_0^z \rho_0(z)\,dz.$$

For this purpose, we assume that $p = p_0 + p^*$, $\rho = \rho_0 + \rho^*$, $U_1 = U_1^*$, $U_2 = U_2^*$, $W = W^*$, and write the equations for p^*, ρ^*, U_1^*, U_2^*, and W^* (in what follows, the asterisk is omitted):

$$\begin{aligned}
&\rho_0\frac{\partial U_1}{\partial t} + \frac{\partial p}{\partial x} = Q_x + F_x = S_x,\\
&\rho_0\frac{\partial U_2}{\partial t} + \frac{\partial p}{\partial x} = Q_y + F_y = S_y,\\
&\rho_0\frac{\partial W}{\partial t} + \frac{\partial p}{\partial z} = Q_z + F_z = S_z,\\
&\rho_0\left[\frac{\partial U_1}{\partial x} + \frac{\partial U_2}{\partial y} + \frac{\partial W}{\partial t}\right] = R + M = \Phi,\\
&\frac{\partial \rho}{\partial t} + W\frac{\partial \rho_0}{\partial z} = T.
\end{aligned} \tag{1.6}$$

Here, the right sides Q_x, Q_y, Q_z, R, and T contain the terms attributable to the rotation of the medium, the viscosity and compressibility of the medium, and nonlinearity, respectively:

$$Q_x = -\rho\frac{\partial U_1}{\partial t} - (\rho + \rho_0)\left[U_1\frac{\partial U_1}{\partial x} + U_2\frac{\partial U_1}{\partial y} + W\frac{\partial U_1}{\partial t} + fU_2\right] + \nu\left[\Delta_3 U + \frac{\partial D}{\partial x}\right],$$

$$Q_y = -\rho\frac{\partial U_2}{\partial t} - (\rho + \rho_0)\left[U_1\frac{\partial U_2}{\partial x} + U_2\frac{\partial U_2}{\partial y} + W\frac{\partial U_2}{\partial t} - fU_1\right] + \nu\left[\Delta_3 V + \frac{\partial D}{\partial y}\right],$$

$$Q_z = -\rho\frac{\partial W}{\partial t} - (\rho + \rho_0)\left[U_1\frac{\partial W}{\partial x} + U_2\frac{\partial W}{\partial y} + W\frac{\partial W}{\partial z} + fU_2\right] + \nu\left[\Delta_3 W + \frac{\partial D}{\partial z}\right],$$

$$R = -\frac{1}{c^2}\left[\frac{\partial p}{\partial t} + U_1\frac{\partial p}{\partial x} + U_2\frac{\partial p}{\partial y} + W\frac{\partial p}{\partial z}\right] - \rho D,$$

$$T = -U_1\frac{\partial \rho}{\partial x} - U_2\frac{\partial \rho}{\partial y} - W\frac{\partial \rho}{\partial z} + \frac{1}{c^2}\left[\frac{\partial p}{\partial t} + U_1\frac{\partial p}{\partial x} + U_2\frac{\partial p}{\partial y} + W\frac{\partial p}{\partial z}\right],$$

$$D = \frac{\partial U_1}{\partial x} + \frac{\partial U_2}{\partial y} + \frac{\partial W}{\partial z}.$$

On the surface $z = 0$, the internal gravity waves produce only small disturbances ζ, this is why after the linearization of the boundary conditions, we have [1, 2, 5, 9]:

$$z = 0: \ W = \frac{\partial \zeta}{\partial t}, \qquad p(x, y, t) = \zeta(x, y, t) g\rho_0(0).$$

From this, we obtain:

$$z = 0: \ \frac{\partial p}{\partial t} - Wg\rho_0(0) = 0. \tag{1.7}$$

Eliminating the variables U_1, U_2, ρ, and p from system (1.6), for the vertical velocity component we obtain the standard equations of internal gravity waves with a certain non-zero right-hand side:

$$\frac{\partial^2}{\partial t^2}\Delta_3 W + N^2(z)\Delta W = Z,$$

$$Z = \frac{N^2(z)}{g}\frac{\partial^3}{\partial z\partial t^2} - \frac{1}{\rho_0}\frac{\partial^2}{\partial z\partial t}\left[\frac{\partial S_x}{\partial x} + \frac{\partial S_y}{\partial y} - \frac{\partial \Phi}{\partial t}\right] + \frac{\Delta_h}{\rho_0}\frac{\partial S_z}{\partial t} - \frac{g}{\rho_0}\Delta T, \tag{1.8}$$

$$N^2(z) = -\frac{g}{\rho_0}\frac{d\rho_0}{dz},$$

where $N^2(z)$ is the Brunt–Vaisala frequency. The boundary condition on the bottom $z = -H$ retains form (1.2) or (1.3). Applying the operator Δ to expression (1.7) and representing ΔP_t in terms of W, we obtain the condition at $z = 0$:

$$\frac{\partial W^3}{\partial z\partial t^2} - g\Delta W = \frac{1}{\rho_0}\left[\frac{\partial^2 \Phi}{\partial t^2} - \frac{\partial^2 S_x}{\partial t\partial x} - \frac{\partial S_y}{\partial t\partial y}\right]. \tag{1.9}$$

In Eq. (1.8) and boundary condition (1.9), the right sides are the sums of the terms depending on the external sources (the body force **F** and the mass source density M) and small corrections which take into account the viscosity, compressibility, and rotation of the medium, and also corrections attributable to the Boussinesq approximation, which are of the order of $N^2/g \ll 1$ [3, 5, 9].

2. ESTIMATE OF THE APPLICABILITY LIMITS OF A LINEAR APPROXIMATION

Equation (1.8) is convenient for the subsequent estimate of these corrections using a perturbation method. The propagation of internal gravity waves in the presence of small additives, attributable to, for example, nonlinear terms, is described by the equation [3, 10]:

$$\frac{\partial^2}{\partial t^2}\Delta_3 W + N^2(z)\Delta W = \varepsilon P(W), \tag{2.1}$$

where the expression for P will include only the terms which take into account the nonlinearity of the original system of hydrodynamic equations [3, 10]. The parameter ε, equal to the ratio of the particle velocity to the phase velocity of the internal gravity waves, shows that these terms are small [3, 8–10]. Below, we will use the formal expansion of the solution of Eq. (2.1) in power series of the parameter ε. We will now study how the correction εP affects the propagation of a single mode of internal gravity waves. To simplify the calculations, we will consider only the plane-flow case, i.e. the flow independent of the coordinate y ($U_2 = 0$). In what follows we will denote $U_1 = U$, then the problem can be formulated as follows. Let the correction εP set in action at $t = 0$:

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 W}{\partial z^2} + \frac{\partial^2 W}{\partial x^2}\right) + N^2(z)\frac{\partial^2 W}{\partial x^2} = \varepsilon\Theta(t)P(W), \tag{2.2}$$

where $\Theta(t) = 0$ for $t < 0$ and $\Theta(t) = 1$ for $t > 0$. The right-hand side of this equation P takes the form:

$$P = \frac{\partial^2}{\partial x\partial t}\left(W\left(\frac{\partial^2 U}{\partial z^2} + \frac{\partial^2 U}{\partial x^2}\right) - U\left(\frac{\partial^2 W}{\partial z^2} + \frac{\partial^2 W}{\partial x^2}\right) + \rho\left(\frac{\partial^2 U}{\partial t\partial z} - \frac{\partial^2 W}{\partial t\partial x}\right)\right.$$
$$\left. + \frac{\partial\rho}{\partial z}\frac{\partial U}{\partial t} - \frac{\partial\rho}{\partial x}\frac{\partial W}{\partial t}\right) + g\frac{\partial^2}{\partial x^2}\left(U\frac{\partial\rho}{\partial x} + W\frac{\partial\rho}{\partial z}\right),$$

where ρ is the density disturbance scaled to a certain typical unperturbed density value ρ_0^*. For the further analysis, for $t < 0$ we may use the solution of Eq. (2.2) without the right side in the form of a fundamental wave mode [10] $W_0 = A\varphi_n(z, k)\cos(\omega_n(k)t - kx)$, where $\omega_n(k)$ and $\varphi_n(z, k)$ are the dispersion curves and the normalized eigenfunctions of the main vertical spectral problem for internal gravity waves [1, 2, 5]

$$\frac{\partial^2\varphi_n(z, k)}{\partial z^2} + k^2\left(\frac{N^2(z)}{\omega_n^2(k)} - 1\right)\varphi_n(z, k) = 0,$$
$$\varphi_n(z, k) = 0, \qquad z = 0, -H. \tag{2.3}$$

Then, the horizontal velocity U_0 and the density disturbance ρ_0 corresponding to zero approximation of W_0 take the form:

$$U_0 = \frac{A}{k}\frac{\partial\varphi_n(z, k)}{\partial z}\sin(\omega_n(k)t - kx),$$
$$\rho_0 = \frac{AN^2(z)}{\omega_n(k)g}\varphi_n(z, k)\sin(\omega_n(k)t - kx). \tag{2.4}$$

For $t > 0$, the solution of (2.2) is sought in the form of a power series with respect to ε: $W = W_0 + \varepsilon W_1 + \varepsilon^2 W_2 + \ldots$ [3, 10]. Then, for the function W_1, we obtain the equation

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 W_1}{\partial z^2} + \frac{\partial^2 W_1}{\partial x^2}\right) + N^2(z)\frac{\partial^2 W_1}{\partial x^2} = P(W_0). \tag{2.5}$$

The solution of Eq. (2.2) coinciding with W_0 for $t < 0$ is continuous, together with its derivative with respect to t. Accordingly, at $t = 0$ the function W_1 and its derivative with respect to t vanish: $W_1 = \partial W_1/\partial t =$

0. Since in real ocean conditions, only the first modes are, as a rule, excited [5, 9], in what follows we will consider the first wave mode. Then, the right side of Eq. (2.5) takes the form:

$$P(W_0) = A^2 2k\sin(2\omega_1(k)t - 2kx)\left[\frac{N^2(z)\omega_1(k)}{gk}\left(\varphi_1(z,\,k)\frac{\partial^2\varphi_1(z,\,k)}{\partial z^2} + \left(\frac{\partial\varphi_1(z,\,k)}{\partial z}\right)^2\right) - \frac{2N^2(z)k\omega_1(k)(\varphi_1(z,\,k))^2}{g} - \frac{\omega_1(k)}{k}\left(\frac{\partial\varphi_1(z,\,k)}{\partial z}\frac{\partial^2\varphi_1(z,\,k)}{\partial z^2} - \varphi_1(z,\,k)\frac{\partial^3\varphi_1(z,\,k)}{\partial z^3}\right) + \frac{\omega_1(k)\varphi_1(z,\,k)}{gk}\frac{\partial\varphi_1(z,\,k)}{\partial z}\frac{\partial N^2(z)}{\partial z} - \frac{k}{\omega_1(k)}\left(\frac{\partial\varphi_1(z,\,k)}{\partial z}\right)^2\frac{\partial N^2(z)}{\partial z}\right] = \Phi(z,\,k)\sin(2\omega_1(k)t - 2kx). \quad (2.6)$$

We seek the solution of Eq. (2.5) in the form of series of the eigenfunctions of problem (2.3):

$$W_1 = \sin(2\omega_1(k)t - 2kx)\sum_{i=1}^{\infty} d_i\varphi_i(z,\,2k). \quad (2.7)$$

The left side of Eq. (2.5) can be represented in the form:

$$\Phi(z,\,k) = N^2(z)\sum_{i=1}^{\infty} c_i\varphi_i(z,\,2k), \qquad c_i = \int_{-H}^{0}\Phi(z,\,k)\varphi_i(z,\,2k)\,dz. \quad (2.8)$$

Substituting (2.7) and (2.8) in (2.6), we obtain

$$d_i = \frac{c_i\omega_i^2(2k)}{4k^2(4\omega_1^2(k) - \omega_i^2(2k))}.$$

Taking the initial conditions into account, we have:

$$W_1 = \sum_{i=1}^{\infty} d_i\varphi_i(z,\,2k)\left[\sin(2\omega_1(k)t - kx) - \frac{\omega_i(2k)\sin^2(2kx) + 2\omega_1(k)\cos^2(2kx)}{\omega_i(2k)} \times \sin(2\omega_1(k)t - 2kx) - \frac{\sin(4kx)}{2\omega_i(2k)}(2\omega_1(k) - \omega_i(2k))\cos(2\omega_1(k)t - 2kx)\right]. \quad (2.9)$$

From (2.9) it is clear that the term with the multiplier d_1 gives the largest contribution to W_1. We will compare the correction W_1 with the unperturbed solution W_0. For this purpose, we replace the first term in (2.9) by the secular (resonant) term (assuming that $\omega_1(2k) = 2\omega_1(k)$ [1, 2]), which only increases the value of W_1,

$$W_1^* = \frac{a_1\omega_1(2k)}{8k^2}\varphi_1(z,\,2k)t\cos(\omega_1(2k)t - 2kx), \quad (2.10)$$

and estimate the time required for W_1 to become comparable with W_0. To calculate the coefficient a_1, we may assume that $\varphi_1(z,\,k) \approx \varphi_1(z,\,2k)$, which is valid for fairly small values of k [1, 2, 5]. As a result, we obtain:

$$a_1 = 2A^2\left(-\frac{\omega_1(k)}{g}\int\limits_{-H}^{0} N^2(z)\varphi_1(z,\,k)\left(\frac{\partial\varphi_1(z,\,k)}{\partial z}\right)^2 dz \,-\, 2k^2\omega_1(k)\omega g\int\limits_{-H}^{0} N^2(z)(\varphi_1(z,\,k))^3\,dz\right.$$

$$\left. +\,\frac{6k^2}{\omega_1(k)}\int\limits_{-H}^{0} N^2(z)(\varphi_1(z,\,k))^2\frac{\partial\varphi_1(z,\,k)}{\partial z}dz \,-\, 3\omega_1(k)k^2\int\limits_{-H}^{0}(\varphi_1(z,\,k))^2\frac{\partial\varphi_1(z,\,k)}{\partial z}dz\right). \quad (2.11)$$

To use formulas (2.10) and (2.11), we will consider a layer of a stratified medium with the depth $H = 100$ m and $N(z) = \text{const} = 0.01\ \text{s}^{-1}$, then $\varphi_1 = \sqrt{2}\sin(\pi z/H)/N\sqrt{H}$. We assume that $k = 0.02\ \text{m}^{-1}$, then the wavelength $\lambda \sim 300$ m and $A = 10^{-4}\ \text{m}^{-4}\ \text{m}^{3/2}\text{s}^{-2}$, which corresponds to the unperturbed wave velocity amplitude of about 14 cm/s. For the case $N(z) = \text{const}$, the last two terms in (2.11) are zero and the first two integrals can be calculated analytically, then $a_1 = -1.4\times 10^{-13}$. We will estimate the time t required for W_1^* to attain the value of about 5% of W_0: $t = 4k^2A\times 0.05/c_1\omega_1(2k) \approx 1.5\times 10^7$ s. This is much greater than the typical oscillation periods of internal gravity waves in the ocean, amounting to tens of minutes [3, 65, 9].

Summary. For the wavelength range, typical of a real ocean, in studying the dynamics of internal gravity waves a linear approximation can be used. In a similar way, it is possible to estimate the effects of other corrections to the linear theory of generation and propagation of internal gravity waves in stratified media. The results of these estimates indicate that the linear model of wave dynamics is adequate and valid for the corresponding spatial and temporal scales.

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