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Far Fields of Internal Gravity Waves in Stratified Media of Variable Depth

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Abstract. The present paper is devoted to presenting the solution of the problem on the far field of internal gravity waves in a stratified medium whose depth linearly depends on the space variable. Uniform asymptotic representations of solutions are constructed; these representations enable us to describe the far fields of internal gravity waves excited by a moving source and characterize the main specific features of the wave fields near caustics and wave fronts.

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1. INTRODUCTION AND SETTING OF THE PROBLEM

As is well known, an essential influence on the propagation of internal gravity waves in stratified natural media (ocean, atmosphere) is caused by the horizontal inhomogeneity and nonstationarity of these media. To the most typical horizontal inhomogeneitities of a real ocean one can refer the modification of the relief of the bottom, an inhomogeneity of the density field, and the variability of the mean flows. One can obtain an exact analytic solution of this problem (for instance, by using the method of separation of variables) only if the distribution of density and the shape of the bottom are described by rather simple model functions. If the shape of the bottom and the stratification are arbitrary, then one can construct only asymptotic representations of the solution in the near and far zones; however, to describe the field of internal waves between these zones, one needs an accurate numerical solution of the problem [1–6].

Using asymptotic methods, one can consider a wide class of interesting physical problems, including problems concerning the propagation of nonharmonic wave packets of internal gravity waves in diverse nonhomogeneous stratified media under the assumption that the modification of the parameters of a vertically stratified medium are slow in the horizontal direction. From the general point of view, problems of this kind can be studied in the framework of a combination of the adiabatic and semiclassical approximations or by using close approach, for example, ray expansions, see [7–9, 16–17, 20–21]. In particular, the asymptotic solutions of diverse dynamical problems can be described by using the Maslov canonical operator [7, 8, 16, 17] which determines the asymptotic behavior of the solutions, including the case of neighborhoods of singular sets composed of focal points, caustics, etc. The specific form of the wave packet can be finally expressed by using some special functions, say, in terms of oscillating exponentials, Airy function, Fresnel integral, Pearcey-type integrals, etc.

The above approaches are quite general and, in principle, enable one to solve a broad spectrum of problems from the mathematical point of view; however, the problem of their practical applications and, in particular, of the visualization of the corresponding asymptotic formulas based on the Maslov canonical operator is still far from completion (see, for example, [11] and [22]), and in some specific problems one can use other schemes to find the asymptotic behavior whose computer realization using software of *Mathematica* type is rather simple. In this paper, using the approaches developed in [3–6], we construct and numerically realize asymptotic solutions of the problem which is formulated as follows.

Consider a layer of a stratified medium with the Brent-Väisälä frequency $N(z)$ and bounded by the surface $z = 0$ and by the bottom $z = H(x, y)$. The source of mass of intensity Q (Q is the volume discharge per second) moves at the depth z_0 uniformly and rectilinearly with the velocity V in the negative direction of the abscissa axis. The system of coordinates under consideration moves together with the source. Then the steady-state field of velocities, in the Boussinesq approximation, satisfies the following linearized system of equations [1–4]:

$$\frac{V^2\partial^2}{\partial\xi^2}\left(\Delta w+\frac{\partial^2}{\partial z^2}w\right)+N^2(z)\Delta w=Q\delta''_{\xi\xi}(\xi)\,\delta(y-y_0)\,\delta'(z-z_0),$$

$$\Delta u+\frac{\partial^2 w}{\partial\xi\,\partial z}=Q\delta'(\xi)\,\delta(y-y_0)\,\delta(z-z_0), \tag{1}$$

$$\Delta v+\frac{\partial^2 w}{\partial y\,\partial z}=Q\delta(\xi)\,\delta'(y-y_0)\,\delta(z-z_0),$$

where

$$\Delta=\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial y^2}\quad\text{and}\quad\xi=x+Vt.$$

On the boundaries of the stratified layer, the following conditions must hold: first, $w=0$ at $z=0$, and

$$w=u\frac{\partial H}{\partial\xi}+v\frac{\partial H}{\partial y}\quad\text{for}\quad z=H(\xi,y). \tag{2}$$

Below we consider the case of $N(z)=\text{const}$ and use the profile of the bottom which is given by a smooth function $H(y)$,

$$H(y)=\begin{cases}\beta y & \text{for}\quad y_1\leqslant y\leqslant y_2,\ y_1>0,\\ \beta y_1 & \text{for}\quad y<y_1,\\ \beta y_2 & \text{for}\quad y>y_2,\end{cases}$$

where y_2 is sufficiently large. The source moves along the line $z=z_0, y=y_0$ $(y_1<y_0<y_2$, $0<z_0<\beta y_0)$. Introduce the horizontal scale,

$$L_y=L_\xi=\frac{V}{N}\frac{\pi}{\beta},$$

and the vertical scale $L_z=\beta y_0/\pi$. We also consider the parameter $\lambda=\pi/\beta$, assuming in what follows that $\lambda\gg 1$. Then we obtain the following relationships among the dimensionless coordinates (ξ^N,y^N,z^N) and the dimensional ones (ξ,y,z):

$$\xi^N=\xi N/\lambda V,\quad y^N=yN/\lambda V,\quad z^N=zN/Vy_0^N,\quad H^N(y)=\pi y/y_0,$$
$$(U_1,U_2,W)=(u,v,w)V^2/QN^2.$$

Note that $y^N=1/M$, where M is the Mach number, $M=V/c_{\max}$, where $c_{\max}$ stands for the maximum of the group velocity of the first (leading) mode of the internal gravity waves,

$$c_{\max}=\frac{NH(y)}{\pi}=Vy^N.$$

Thus, the domain $y^N<1$ corresponds to $M>1$, that is, the speed of the source exceeds $c_{\max}$. The domain $y^N>1$ corresponds to $M<1$, and $c_{\max}$ exceeds the speed of the motion of the source V. Assume that $y_0^N<1$, i.e., the speed of the source is greater than $c_{\max}$.

In the dimensionless coordinates (here and below, we omit the superscript N), the system of equations (1) and the boundary conditions (2) looks as follows:

$$\frac{\partial^2}{\partial\xi^2}\left(\frac{1}{\lambda^2}\Delta+\frac{1}{y_0^2}\frac{\partial^2}{\partial z^2}\right)W+\Delta W=\frac{1}{\lambda^2y_0^2}\delta''_{\xi\xi}(\xi)\,\delta(y-y_0)\,\delta'_z(z-z_0),$$

$$\frac{1}{\lambda}\Delta U_2 + \frac{1}{y_0}\frac{\partial^2 W}{\partial y \partial z} = \frac{1}{\lambda^2 y_0}\delta(\xi)\,\delta'(y-y_0)\,\delta(z-z_0), \tag{3}$$

$$\frac{1}{\lambda}\Delta U_1 + \frac{1}{y_0}\frac{\partial^2 W}{\partial \xi \partial z} = \frac{1}{\lambda^2 y_0}\delta'(\xi)\,\delta(y-y_0)\,\delta(z-z_0),$$

$$W = 0 \quad \text{for} \quad z = 0, \tag{4}$$

and

$$W = \pi U_2/\lambda \quad \text{for} \quad z = H(y).$$

To these conditions, one must add the condition that the functions W, U_1, and U_2 decay at infinity.

2. ANSATZ FOR THE INCIDENT WAVE

Since the boundary condition (4) does not contain the component of the horizontal velocity U_1, it follows that system (3) is decomposed into the first two equations and the separate third equation. After solving the first two equations, one can find the function U_1 from the third equation. Therefore, let us construct a solution of the first two equations for the components W and U_2 (we shall omit the subscript 2). Following [2, 7, 13], we seek a particular asymptotic solution of these two equations of system (3) in the form

$$W = \int F(z,y,\omega)\exp(i\lambda(\omega\xi - S(y,\omega)))\,d\omega, \qquad U = \int \Psi(z,y,\omega)\exp(i\lambda(\omega\xi - S(y,\omega)))\,d\omega, \tag{5}$$

where

$$F = F_0(z,y,\omega) + \frac{i}{\lambda}F_1(z,y,\omega) + O\Big(\frac{1}{\lambda^2}\Big) \quad \text{and} \quad \Psi = i\Psi_0(z,y,\omega) + O\Big(\frac{1}{\lambda}\Big).$$

Note that the functions W and U depend on ξ only by means of the stage of elementary solutions whose superpositions define the solutions W and U. Thus, the problem is reduced to finding the functions $F(z,y,\omega)$ and $S(y,\omega)$. Substituting W and U given by (5) into the first two equations (3) and the boundary conditions (4) and equating the terms at like powers of λ, we obtain the eikonal equation,

$$\omega^2 + \left(\frac{\partial S_n(y,\omega)}{\partial y}\right)^2 = k_n^2(\omega,y),$$

the dispersion relation

$$k_n^2(\omega,y) = \frac{\omega^2 n^2}{(1-\omega^2)y^2},$$

and an eigenfunction

$$F_{0n}(z,y,\omega) = c_n(y,\omega)\sin\frac{nzy_0}{y},$$

where $c_n(y,\omega)$ stands for an arbitrary function independent of z. The complete solution is represented as the sum of n special solutions ("vertical modes"). In what follows, we consider the first mode only, $n = 1$ (and omit the subscript 1).

The function $c(y,\omega)$ is defined by the corresponding conservation law along the characteristics (rays) of the eikonal equation and from the locality principle (cf. [20, 21]), and this function is of the form

$$c(y,\omega) = c(y_0,\omega)\frac{y_0}{y}\sqrt[4]{\frac{1-(1-\omega^2)y_0^2}{1-(1-\omega^2)y^2}}.$$

The value of $c(y_0,\omega)$ can be determined by using the locality principle, i.e., using the solution of the problem with the constant depth of the bottom, $H = H(y_0)$, namely,

$$c(y_0,\omega) = \frac{2i\omega\,\sin z_0}{\pi\sqrt{1-\omega^2}\sqrt{1-(1-\omega^2)y_0^2}}$$

(see [3–6]). We finally obtain

$$F_0(z,y,\omega)=\frac{2i}{\pi}\frac{y_0}{y}\frac{\omega}{\sqrt{1-\omega^2}}\frac{\sin z_0\ \sin\left(\frac{zy_0}{y}\right)}{\sqrt[4]{1-(1-\omega^2)y^2}\sqrt[4]{1-(1-\omega^2)y_0^2}}.$$

Further, let us consider the eikonal equation,

$$\left(\frac{\partial S(y,\omega)}{\partial y}\right)^2=\frac{\omega^2}{(1-\omega^2)y^2}-\omega^2,$$

whose solution is of the form

$$S(y,\omega)=\int_{y_0}^{y}\frac{\omega\sqrt{1-y^2\alpha}}{y\sqrt{\alpha}}dy=\frac{\omega}{\sqrt{\alpha}}\Bigg(\left(\sqrt{1-y^2\alpha}-\ln\left(1+\sqrt{1-y^2\alpha}\right)+\ln y\right) - \left(\sqrt{1-y_0^2\alpha}-\ln\left(1+\sqrt{1-y_0^2\alpha}\right)+\ln y_0\right)\Bigg), \tag{6}$$

where

$$\alpha=1-\omega^2.$$

Then the field of the incident wave (in the domain $y_0<y<1$) looks as follows [9, 12, 13]:

$$W=\int_0^1\frac{4}{\pi}\frac{y_0}{y}\frac{\omega}{\sqrt{1-\omega^2}}\frac{\sin z_0\ \sin\frac{zy_o}{y}}{\sqrt[4]{1-y^2(1-\omega^2)}\sqrt[4]{1-y_0^2(1-\omega^2)}}\cos\left(\lambda\left[\omega\xi-S(y,\omega)\right]+\frac{\pi}{2}\right)d\omega. \tag{7}$$

For large values of λ, integral (7) can be studied by using the method of stationary phase. Introduce a family of rays which is defined as the set of points at which the phase

$$\Phi(\omega,\xi,y)=\lambda\left(\omega\xi-s(y,\omega)\right)+\frac{\pi}{2}$$

is stationary, i.e.,

$$\frac{\partial\Phi(\omega,\xi,y)}{\partial\omega}=0\quad\text{or}\quad\xi(y,\omega)=\frac{\partial S(y,\omega)}{\partial\omega}.$$

Differentiating (6) with respect to ω, we obtain the family of rays

$$\xi(y,\omega)=\frac{\alpha\sqrt{1-y^2\alpha}-\ln\left(1+\sqrt{1-y^2\alpha}\right)+\ln y-\alpha\sqrt{1-y_0^2\alpha}+\ln\left(1+\sqrt{1-y_0^2\alpha}\right)-\ln y_0}{\alpha^{3/2}}. \tag{8}$$

Expression (8) defines a one-parameter family of ascending rays on the plane ξ,y that issue from the point $\xi=0,y=y_0$ with the parameter α (or ω). For a fixed ω, we obtain a particular ray. Obviously, this expression describes the behavior of the ray only before the turning point

$$y_*=\frac{1}{\sqrt{\alpha}},\qquad \xi_*=\xi(y_*,\alpha)$$

(i.e., for $\xi\leqslant\xi_*$) [9, 12, 13].

3. ANSATZ FOR THE REFLECTED WAVE

The ray reflected by refraction ($\xi \geqslant \xi_*$) is constructed as follows. Let us first define the eikonal $S_1(y,\omega)$ of the reflected ray,

$$\begin{aligned} S_1(y,\omega) &= S(y_*,\omega) - \int_{y_*}^{y} \frac{\omega\sqrt{1-y^2\alpha}}{y\sqrt{\alpha}}dy \\ &= \frac{\omega}{\sqrt{\alpha}}\Big(\Big(-\sqrt{1-y^2\alpha}+\ln\left(1+\sqrt{1-y^2\alpha}\right)-\ln y-\ln\alpha\Big) \\ &\quad -\left(\sqrt{1-y_0^2\alpha}-\ln\left(1+\sqrt{1-y_0^2\alpha}\right)+\ln y_0\right)\Big). \end{aligned} \tag{9}$$

In this case, on the entire domain $0<y<1$, the reflected field W_1 looks as follows:

$$W_1 = \int_0^1 \frac{4}{\pi}\frac{y_0}{y}\frac{\omega}{\sqrt{1-\omega^2}}\frac{\sin z_0\ \sin\frac{zy_0}{y}}{\sqrt[4]{1-y^2(1-\omega^2)}\sqrt[4]{1-y_0^2(1-\omega^2)}}\cos\left(\lambda\left(\omega\xi - S_1(y,\omega)\right)+\pi\right)d\omega, \tag{10}$$

that is, as compared with W, the eikonal $S_1(y,\omega)$ of the reflected wave is taken instead of the eikonal $S(y,\omega)$ of the incident wave and the phase shift is taken into account, $(+\pi/2)$, which occurs due to the reflection of the ray [7–9, 12, 13]. The ray reflected due to the refraction is centroid similarly to the incident ray,

$$\xi_1(y,\alpha) = \frac{\partial S_1}{\partial\omega}, \tag{11}$$

which gives

$$\begin{aligned} \xi_1(y,\alpha) = \alpha^{-3/2}\Big(&-\alpha\sqrt{1-y^2\alpha}+\ln\left(1+\sqrt{1-y^2\alpha}\right) \\ &-\ln y-\alpha\sqrt{1-y_0^2\alpha}+\ln\left(1+\sqrt{1-y_0^2\alpha}\right)-\ln y_0-\ln\alpha\Big). \end{aligned}$$

It follows from (8) and (11) that the incident and reflected rays are symmetric for $y_0<y<1$ with respect to the line $\xi=\xi^*$. In the domain $0<y<y_0$, the reflected field is described by the integral (11), and the field of descending incident wave is defined by (7), where S is replaced by $-S$.

4. SIMPLIFICATION OF THE ASYMPTOTIC FORMULAS FOR THE INCIDENT AND REFLECTED WAVES

Let us proceed with estimating the fields given in an integral form by using the example of the incident wave (7) and applying the method of stationary phase ($\lambda \gg 1$) (see, for example, [10]):

$$W(\xi,y) = \frac{B(\omega,y,y_0,z,z_0)}{\sqrt{\lambda\frac{\partial\xi(y,\omega)}{\partial\omega}}}\cos\left[\lambda\left(\omega\xi - S(y,\omega)\right)+\frac{\pi}{4}\right], \tag{12}$$

$$B(\omega,y,y_0,z,z_0) = \frac{4\sqrt{2}y_0\omega\ \sin z_0\ \sin\frac{zy_0}{y}}{\sqrt{\pi}\,y\sqrt{1-\omega^2}\sqrt[4]{1-y^2(1-\omega^2)}\sqrt[4]{1-y_0^2(1-\omega^2)}},$$

where the function $\xi(y,\omega)$ is defined in (8). In expression (12), the frequency $\omega=\omega(\xi,y)$ is defined implicitly, for any fixed ξ and y, as a solution of the equation $\xi=\xi(y,\omega)$. Thus, to obtain, for example, the picture of the incident wave for a fixed y as a function of ξ, it is necessary to resolve the defining equation $\xi=\xi(y,\omega)$ with respect to ω for any value of ξ and to substitute the value thus obtained into (12). However, the construction of this field can be substantially simplified, namely, one can regard (12) as a function of the variable ω only for a fixed y, i.e., $W\left(\xi(y,\omega),y\right)$. In this case, together with the equation of the ray $\xi = \xi(y,\omega)$, one can obtain the desired dependence $W(\xi, y)$ in a parametric way, because all functions entering expression (12) are given explicitly.

Similarly, by the method of stationary phase, in a parametric way, one can construct the picture of a field for the wave reflected due to the refraction, $W_1(\xi, y)$, namely,

$$W_1(\omega, y) = -\frac{B(\omega, y, y_0, z, z_0)}{\sqrt{\lambda \frac{\partial \xi_1(y,\omega)}{\partial \omega}}} \cos\left[\lambda\left(\omega\xi_1(y,\omega) - S_1(y,\omega)\right) + \frac{3\pi}{4}\right], \tag{13}$$

with

$$\xi = \xi_1(y,\omega),$$

and also for the incident field $W_2(\xi, y)$ formed by ascending rays,

$$W_2(\omega, y) = -\frac{B(\omega, y, y_0, z, z_0)}{\sqrt{\lambda\left(-\frac{\partial \xi(y,\omega)}{\partial \omega}\right)}} \cos\left[\lambda\left(-\omega\xi(y,\omega) + S(y,\omega)\right) + \frac{\pi}{4}\right], \tag{14}$$

with

$$\xi = -\xi(y,\omega).$$

Note that the radicand in the denominator (14) is positive, because the upper limit of integration is less than the corresponding lower limit in the corresponding expression for the eikonal. The amplitudes of the fields thus obtained decay for large ξ by the rule of $\xi^{-1/2}$. This can readily be seen by considering the equation of a ray, for example, for the incident wave (7). To large values of ξ there correspond small quantities of α (for a fixed y), and then $\xi \sim \alpha^{-3/2}$ and $\xi' \sim \alpha^{-5/2}$. Substituting these estimates into the amplitude of the incident wave, one can obtain the desired result.

5. FIELDS IN A NEIGHBORHOOD OF CAUSTICS

The wave domain of solutions in the zone of supercritical velocities is bounded by the line $y = 1$ and by the caustic (the envelope) of the incident and ascending rays. Therefore, first of all, let us describe the behavior of a caustic which is defined from the solution of the system [2, 9, 12, 13]

$$\frac{\partial \xi(y,\alpha)}{\partial \omega} = 0 \tag{15}$$

with

$$\xi = \xi(y,\alpha).$$

Here, since $\alpha = 1-\omega^2$, it follows that the first equation in (15) is decomposed into two equations, namely, $\omega = 0$ and $\frac{\partial \xi(y,\alpha)}{\partial \alpha} = 0$. Then we obtain the envelope in the proper sense for the system given by the equation

$$\frac{\partial \xi(y,\alpha)}{\partial \alpha} = 0 \tag{16}$$

together with

$$\xi = \xi(y,\alpha),$$

and a curve similar to a "beak" and formed by the incident and the reflected ray for $\omega = 0$ ($\alpha = 1$), namely, $\xi = \xi(0, y)$ and $\xi = \xi_1(0, y)$, where $\xi(\omega, y)$ and $\xi_1(\omega, y)$ are defined in (8) and (11). For the values of y close to 1, the equation of the "beak" is a semicubical parabola with the cusp at $\xi = \xi_0$ and $y = 1$, namely,

$$(\xi - \xi_0)^2 = \frac{8}{9}(1-y)^3, \qquad \xi_0 = \ln\left(1 + \sqrt{1-y_0^2}\right) - \sqrt{1-y_0^2} - \ln y_0.$$

The first equation in (16) can also be represented in the form

$$\sqrt{1-y^2\alpha}=\frac{1-\alpha}{\varphi(y,\alpha)}$$

and

$$\varphi(y,\alpha)=\frac{\alpha-1}{\sqrt{1-y_0^2\alpha}}-3\left(\ln y-\ln y_0-\ln\left(1+\sqrt{1-y^2\alpha}\right)+\ln\left(1+\sqrt{1-y_0^2\alpha}\right)\right).$$

When describing the field of vertical velocity of the internal gravity waves on the domain $y>1$, one must take into account the turning point

$$\omega_*=\frac{\sqrt{y^2-1}}{y},$$

which is a branching point on the complex plane ω (see [7–9, 12, 13]). At this point, the amplitude of the integrand in (5) becomes infinite. Thus, the interval of integration with respect to ω from 0 to 1 is decomposed into two parts; the first of them lasts from 0 to ω_*, whereas the other from ω_* to 1. The second interval of integration corresponds to the zone to the right of the caustic (real rays), whereas the first interval of integration corresponds to the zone to the left of the caustic (complex rays). In this case, the incident and the reflected fields are defined from (7) and (10), where the lower limit of integration is ω_*. In order to find the integrand on the first interval, i.e., the field of the so-called penetrating waves, we analytically continue the integrand of the incident wave with respect to ω through the lower half-plane (or the reflected wave through the upper one) [7–9, 12, 13]. Anyway, we obtain the following expression for the penetrating wave:

$$W_2=\mathrm{Re}\int_0^{\omega_*}\frac{4e^{i\frac{3\pi}{4}}y_0\omega\ \sin z_0\ \sin\frac{zy_0}{y}}{\pi y\sqrt{1-\omega^2}\,(y^2\alpha-1)^{1/4}\,(1-y_0^2\alpha)^{1/4}}\ \exp(i\lambda(\omega\xi-S_2(\omega,y)))\,d\omega,$$

$$S_2(\omega,y)=\frac{\omega}{\sqrt{1-\omega^2}}\biggl(-i\sqrt{y^2\alpha-1}-\ln\left(1-i\sqrt{y^2\alpha-1}\right)$$
$$+\ln y-\sqrt{1-y_0^2\alpha}+\ln\left(1+\sqrt{1-y_0^2\alpha}\right)-\ln y_0\biggr).$$

Thus, the total field W_c consists of three summands, namely, $W_c=W+W_1+W_2$. Let us write out the integrand f_c which enables one, according to (5), compute the total field of the vertical component of the velocity of the internal gravity waves. We have

$$\begin{aligned}
f_c&=f+f_1\quad\text{for}\quad\omega>\omega_*,\\
f_c&=f_2\qquad\text{for}\quad\omega<\omega_*,\\
f&=Re\frac{b(\omega,y)}{(1-y^2\alpha)^{\ 1/4}}\ \exp\left(i\ \left(\lambda(\omega\xi-S(\omega,y))+\frac{\pi}{2}\right)\right),\\
f_1&=\mathrm{Re}\,\frac{b(\omega,y)}{(1-y^2\alpha)^{\ 1/4}}\ \exp(i\ (\lambda(\omega\xi-S_1(\omega,y))+\pi)),\\
f_2&=\mathrm{Re}\,\frac{b(\omega,y)}{(y^2\alpha-1)^{\ 1/4}}\ \exp\left(i\ \left(\lambda(\omega\xi-S_2(\omega,y))+\frac{3\pi}{4}\right)\right),\\
b\,(\omega,y)&=\frac{4y_0\omega\ \sin z_0\ \sin\frac{zy_0}{y}}{\pi y\sqrt{1-\omega^2}\,(1-y_o^2\alpha)^{\ 1/4}}.
\end{aligned}\tag{17}$$

The functions f, f_1, and f_2 are elementary solutions of the equation of internal gravity waves for a fixed frequency ω in the geometric optics approximation, or elementary WKB solutions [2, 7–9, 12, 13].

6. MATCHING OF ASYMPTOTICS

In what follows, one must find a function $G(\omega, \xi, y)$ which is regular at the point $\omega = \omega_*$ and whose asymptotic behavior coincides, for large values of λ, to the asymptotic behavior of the function f_2 to the left of the turning point and to the asymptotic behavior of $f + f_1$ to the right of the turning point [10]. To this end, we introduce the following function:

$$\Delta(\omega, y) = S(y_*, \omega) - S(y, \omega), \tag{18}$$

i.e.,

$$\Delta(\omega, y) = \frac{\omega}{\sqrt{\alpha}} \left(\ln \frac{1}{\sqrt{\alpha}} - \sqrt{1 - y^2\alpha} + \ln \left(1 + \sqrt{1 - y^2\alpha} \right) - \ln y \right),$$

which is the phase increment from the point y to the turning point

$$y_* = \frac{1}{\sqrt{\alpha}}.$$

One can expand the function $\Delta(\omega, y)$ in a series with respect to the argument αy^2, assuming that this argument is close to 1, and restrict ourselves to the leading term of the expansion,

$$\Delta \approx \frac{\omega}{\sqrt{\alpha}} \frac{\sqrt{1 - y^2\alpha}^3}{3}.$$

However, it seems to be more convenient to analytically continue the function $\Delta(\omega, y)$ to the domain $\omega > \omega_*$ as follows:

$$\Delta(\omega, y) = \frac{i\omega}{\sqrt{\alpha}} \left(\sqrt{y^2\alpha - 1} - \arctan \left(\sqrt{y^2\alpha - 1} \right) \right).$$

Considering the leading term of this expression only and assuming that $\alpha y^2 \sim 1$, we can see that

$$\Delta \approx \Delta_1(\omega, y) = \frac{\omega}{3\sqrt{\alpha}} (1 - y^2\alpha)^{3/2} \qquad \text{for } \omega < \omega_* \tag{19}$$

and

$$\Delta \approx \Delta_2(\omega, y) = \frac{i\omega}{3\sqrt{\alpha}} (y^2\alpha - 1)^{3/2} \qquad \text{for } \omega > \omega_*.$$

Therefore, formula (17) can be represented as

$$\varphi(\omega, \xi) = \lambda \left(\omega\xi - S\left(y, \omega\right) \right) + \frac{3\pi}{4},$$

$$f = b\left(\omega, y\right)\, \exp(i\varphi(\omega, \xi)) \frac{\exp\left(i\, \left(\lambda\Delta_1 - \frac{\pi}{4} \right) \right)}{\sqrt[4]{1 - y^2\alpha}}, \tag{20}$$

$$f_1 = b\left(\omega, y\right)\, \exp(i\varphi(\omega, \xi)) \frac{\exp\left(-i\, \left(\lambda\Delta_1 - \frac{\pi}{4} \right) \right)}{\sqrt[4]{1 - y^2\alpha}},$$

$$f_2 = b\left(\omega, y\right)\, \exp(i\varphi(\omega, \xi)) \frac{\exp(-\lambda\, |\Delta_2|)}{\sqrt[4]{y^2\alpha - 1}}.$$

Expressions (20), together with (19), present the asymptotics of the following function [10, 18, 19]:

$$G^*(\omega, \xi, y) = 2\sqrt{\pi}\, \cos\left(\varphi(\omega, \xi) \right)\, b(\omega, y)\, q^{1/6} Ai \left(q^{2/3} (1 - y^2\alpha) \right), \tag{21}$$

where

$$q = \frac{\lambda\omega}{2\sqrt{1-\omega^2}}$$

and $Ai(x)$ stands for the Airy function. The function G^* is a local asymptotic and describes the solution near the turning point $\omega = \omega_*$ [7–9]. It can readily be seen that the expressions (20) give an asymptotic of the function $G^*(\omega, \xi, y)$. To this end, it is sufficient to use the asymptotic behavior of the Airy function [18, 19],

$$Ai(x) \sim \frac{1}{\sqrt{\pi}x^{1/4}} \cos\left(\frac{2}{3}x^{3/2} - \frac{\pi}{4}\right) \quad \text{as} \quad x \to +\infty$$

and

$$Ai(x) \sim \frac{1}{2\sqrt{\pi}\,|x|^{\ 1/4}} \exp\Big(-\frac{2}{3}\,|x|^{3/2}\Big) \quad \text{as} \quad x \to -\infty.$$

Note that the asymptotic relation (21) works in the zone of subcritical velocities ($y < 1$) for which there are no turning points in the domain of integration.

The uniform asymptotic $G(\omega, \xi, y)$ looks as follows: [10, 18, 19]

$$G(\omega, \xi, y) = \frac{2\sqrt{\pi}\ b(\omega, y)\ \cos\left(\varphi(\omega, y)\right)\left(\frac{3}{2}\lambda\Delta(\omega, y)\right)^{1/6}}{\sqrt{1-y^2\alpha}} Ai\left(\left(\frac{3}{2}\lambda\Delta(\omega, y)\right)^{2/3}\right), \tag{22}$$

where $\Delta(\omega, y)$ is defined in (18).

Note that the uniform asymptotic (22) passes for small arguments of the Airy function to the local asymptotic (21) and for large arguments to the WKB expansion (20) (see [10]). The uniform asymptotic also works in the zone of subcritical velocities ($y < 1$). Obviously, for $y < y_0$, the incident (descending) wave is computed by using other formulas.

The complete field W of a single mode of the vertical velocity of internal gravity waves, in accordance with (5), is obtained by integrating expressions (20), (21), or (22) with respect to ω from 0 to 1.

The uniform asymptotic (22) describes the integrand of the solution for W in the zone $y > y_0$ and for any ω except for a neighborhood of the point $\omega = 0,\ y = 1$ (on the plane ξ, y this is a neighborhood of the cusp, the "beak," of the caustic [7–9, 12, 13]). The argument of the Airy function fails to be an analytic function for small ω (this argument behaves as $\omega^{2/3}$), and the amplitude looks as $\omega^{7/6}$. Let us find the behavior of the integrand for $y = 1$ and for any ω. For an appropriate function coinciding with the uniform Airy function for large ω and satisfying the analyticity for small ω, one can take the following function [18, 19]:

$$F(\omega) = b(\omega, 1)(\lambda\Delta(\omega, 1))^{1/8}\, D\left(1/4, (\lambda\Delta(\omega, 1))^{1/4}\right)/\sqrt{\omega},$$

where the functions

$$D(\nu, x) = \frac{1}{\pi}\,\mathrm{Re}\ \exp(-ix^4)\int_0^\infty \exp(i(t-x^2)^2)t^{-\nu}\,dt$$

can be expressed by using the parabolic cylinder functions [18, 19].

Consider further the function

$$M(\omega, y) = \frac{\sqrt{y}\left(\frac{3}{2}\lambda\Delta(\omega, y)\right)^{1/6}}{\sqrt[4]{1-y^2(1-\omega^2)}} Ai\Big(\Big(\frac{3}{2}\lambda\Delta(\omega, y)\Big)^{2/3}\Big),$$

which coincides (up to amplitude factor) with the function $G(\omega, \xi, y)$ (22) and is a uniform asymptotic, with regard to the turning point, for the following generating equation:

$$\frac{\partial^2 m(y, \omega)}{\partial y^2} + \lambda^2\omega^2\Big(\frac{1}{(1-\omega^2)y^2} - 1\Big)m(y, \omega) = 0, \tag{23}$$

which is regarded as an equation with a single turning point

$$y_* = \frac{1}{\sqrt{1-\omega^2}}$$

(in the domain $y > 0$). Equation (23) is obtained from the first equation of system (3) if we seek a solution in the form

$$W = c(y,\omega)m(y,\omega)\exp(i\lambda\omega\xi)\sin\Big(\frac{zy_0}{y}\Big).$$

We must take into account here that

$$\frac{\partial m}{\partial y} \sim O(\lambda), \qquad \frac{\partial^2 m}{\partial y^2} \sim O(\lambda^2), \qquad \frac{\partial c}{\partial y} \sim O(1).$$

In particular, equation (23) implies the eikonal equation in the WKB approximation. However, the asymptotic $M(\omega, y)$ does not fit for small values of ω (for $\omega \ll \frac{1}{\lambda}$). To find an asymptotic behavior available for small values of ω, consider the exact solution of equation (23). The linearly independent solutions of this equation, m_1 and m_2, can be expressed in terms of the modified Bessel functions,

$$m_1 = \sqrt{y}I_\nu(\lambda\omega y), \qquad m_2 = \sqrt{y}K_\nu(\lambda\omega y),$$

with the subscript

$$\nu(\omega) = \frac{1}{2}\sqrt{\frac{1-(1+4\lambda^2)\omega^2}{1-\omega^2}},$$

which can take both real and imaginary values, in dependence on ω [18, 19]. The Macdonald function

$$m(y,\omega) = A(\omega)\sqrt{y}K_\nu(\lambda\omega y)$$

is an appropriate solution, because it decays for large positive values of y. To find the normalizing factor, one must consider the coincidence of the asymptotics of the functions $M(\omega, y)$ and $m(\omega, y)$ as $y \to +\infty$, namely,

$$M(\omega,y) \sim \frac{\exp(-\lambda\omega y + \lambda\omega\pi/2\sqrt{1-\omega^2})}{2\sqrt{\pi}\sqrt[4]{1-\omega^2}}$$

and

$$m(\omega,y) \sim A(\omega)\frac{\sqrt{\pi}\exp(-\lambda\omega y)}{\sqrt{2\lambda\omega}}.$$

Using these relations, one can prove that

$$A(\omega) = \frac{\sqrt{\lambda\omega}\exp(\lambda\omega\pi/2\sqrt{1-\omega^2})}{\pi\sqrt{2}\sqrt[4]{1-\omega^2}}.$$

With regard to the amplitude factor obtained from the corresponding conservation law along the characteristics (rays) and from the locality principle, one can finally obtain the following asymptotic expression describing the integrand in (5) for any values of ω:

$$M_*(y,\omega) = 2\sqrt{\pi}A(\omega)b(\omega,y)\cos(\varphi(\omega,\xi))K_\nu(\lambda\omega y). \tag{24}$$

Obviously, the complete field W of a separate mode of the vertical velocity for internal gravity waves is obtained, in accordance with (5), by integrating expression (24) with respect to ω from 0 to 1.

7. NUMERICAL REALIZATION OF ASYMPTOTIC FORMULAS

In Fig. 1, we present results of computation of the integrand $F(z, y, \omega)$ in dependence on ω for chosen values of ξ and y, namely, the WKB approximation (20) is given by the dashed line, the

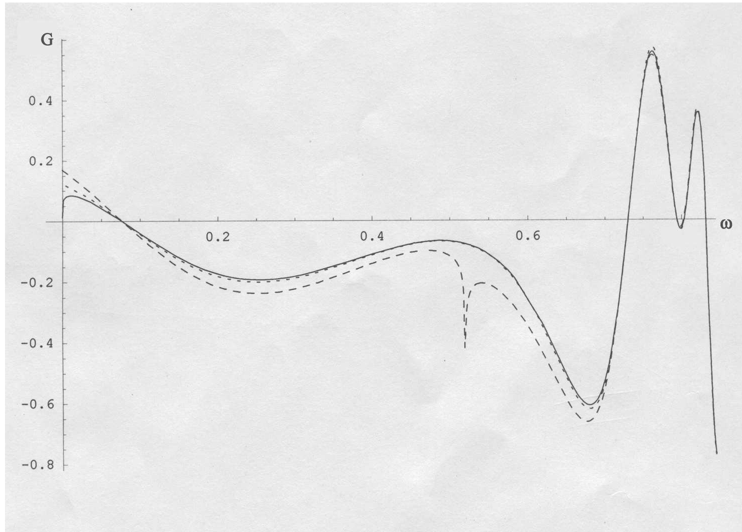

Fig. 1. F

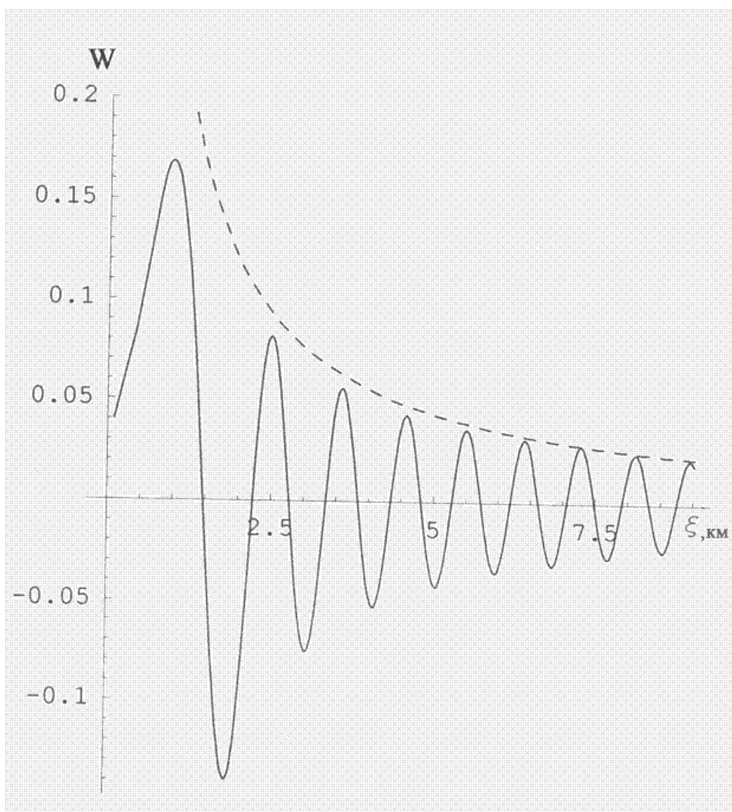

Fig. 2.

Airy uniform asymptotic (22) by the continuous line, and the Macdonald asymptotic function (24) by the dotted line. The dimensionless parameters of computation are

$$y = 1.17, \qquad \xi = 2, \qquad \lambda = 9.5.$$

In Fig. 2, we present the complete wave field W of the vertical component of the velocity in

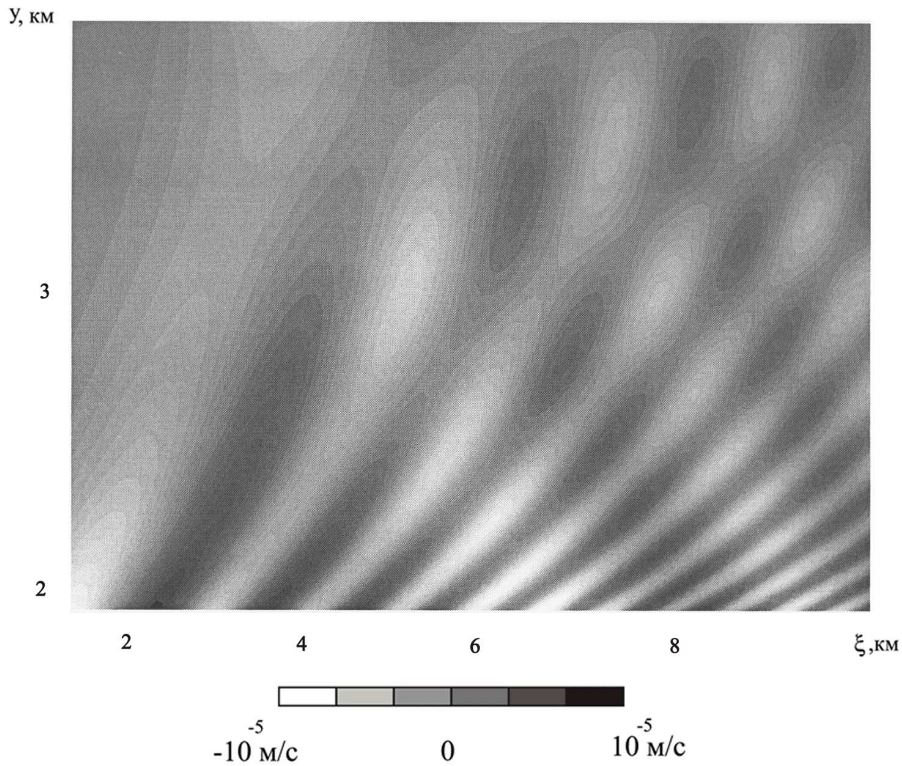

Fig. 3.

dimensional coordinates that is obtained by integrating the Airy uniform asymptotic (22) along the caustic (the continuous line). The main parameters of the computation, which are typical for a real oceanic shelf [14, 15], were chosen as follows: the slope of the bottom $\beta = 0.1$, $V = 2\text{m/s}$, and $y_0 = 500$ meters. The dashed line in this figure represents the law of diminishing amplitudes of the field on the caustic, const $\xi^{-10/9}$, which is obtained from the analysis, using the perturbation method, of the behavior of the integrand in (5). In our case, we have const $= 0.26$ and, as the numerical computations presented in this way show, the amplitude of the wave field is qualitatively correctly described by this dependence indeed. In Fig. 3, we present results of computation of the amplitude-phase picture of the vertical velocity of the first water in the dimensional coordinates. It follows from the numerical results thus presented that, outside the caustic, the wave field is sufficiently small indeed and is not subjected to great many oscillations, whereas the wave picture inside the zone of caustic is a rather complicated system of incident and reflected harmonics.

Thus, in the adiabatic approximation, we have obtained computation formulas for the field of vertical velocity of the separate mode (the first one) in two zones, namely, in the zone of subcritical velocities ($y < 1$) and in the zone of supercritical velocities ($y < 1$).

In the first zone, the field is represented in the form of two integrals which are a superposition of quasiplane waves. The first of them defines the incident wave, and the other represents the reflected wave. The integrals thus obtained are evaluated by the method of stationary phase. Formulas (12) and (13) are the results of these computations.

In the other zone, which includes the caustic, the field is represented by a single integral with respect to the frequency ω. The uniform asymptotic of the integrand is expressed in terms of the Airy function or the Macdonald function of a variable index depending on the frequency ω (formulas (22) and (24)). The asymptotics thus obtained are analyzed numerically.

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